ON THE LINEAR STABILITY OF INVISCID PARALLEL SHEAR FLOWS

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To

M. Subramanian

and

Mihir B. Banerjee

with love.

POST GRADUATE OFFICE
Submitted On 24/9/86

CERTIFICATE

This is to certify that the work embodied in the thesis "ON THE LINEAR STABILITY OF INVISCID PARALLEL SHEAR FLOWS" by M. Subbiah has been carried out under my supervision and has not been submitted elsewhere for a degree or diploma.

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CONTENTS

SYNOPSIS

The present work is devoted to the investigation of the linear stability of inviscid parallel shear flows.

Howard (1961), in one of the most beautiful theorems in hydrodynamic stability of homogeneous shear flows, has shown that the eigenvalues c of unstable modes lie inside a semi-circle, in the upper half of the complex plane, whose diameter is the range of the basic velocity. This result does not incorporate the effect of curvature of the basic velocity profile on the stability of the homogeneous shear flows. But, it is curvature of the basic velocity profile that plays dominant role on the stability or instability of homogeneous shear flows (Rayleigh, 1880; Fjortoft, 1950; Tollmien, 1935; Lin, 1955). Motivated by these arguments, the problem is re-investigated and an instability region depending on the curvature is obtained. This region comes out to be depth dependent as well. In conjunction with Howard's result, it reduces the instability region given by Howard's semicircle.

Instead of a homogeneous fluid, if one considers a horizontally stratified fluid under the action of gravity, then the question. What is the role of curvature on the stability of the flow? does not have a satisfactory answer till today. In fact, the validity or otherwise of the Rayleigh's inflexion point theorem in the presence of stratification has not been

established so far. Motivated by these arguments, the role of curvature of the basic flow profile on the stability of stratified shear flows is studied. Simple generalizations of Hoiland's (1951) criterion and Sattinger's estimate are given. In addition to generalizing the instability region discussed above, it is shown that the complex wave velocity of an arbitrary arbitrary unstable mode lie inside a semiellipse region whose major axis coincides with that of Kochar-Jain's (1979) semiellipse, while its minor axis depends not only on the stratification but also on the curvature of the basic velocity profile.

The work of Howard (1961) show that the complex wave velocities of unstable and their conjugate damped modes lie inside a bounded region in the $c_r c_i$-plane. The wave velocities of singular neutral modes, by their very definition, lie between the minimum and maximum values of the basic velocity. In this work, it is shown that the wave velocities of non-singular neutral modes of the Taylor-Goldstein problem are also bounded.

Stability of parallel shear flow of a compressible fluid is of interest in meteorology. Unfortunately however, there is not much systematic work on the stability characteristics of inviscid compressible parallel shear flows. Following Blumen (1970) and Chimonas (1970), this problem is studied with and without the presence of gravity. Following Blumen gravity is ignored and the basic thermodynamic state is assumed to be

constant. It is shown that shear free basic flow ($U \equiv 0$) supports supersonic waves. Further, it is shown that the complex wave velocity of an unstable subsonic disturbance lie inside a semiellipse type region depending on the Mach number M. Eckart's (1963) semicircle comes out to be a special case of this when $M = 0$. Incidently, this region comes out to be depth as well as wave number dependant. For unbounded flows, a sufficient condition for stability to supersonic disturbances and an estimate for the growth rate of an unstable supersonic disturbances are also given.

For stratified compressible shear flows, it is shown that the instability region for subsonic disturbances is an semiellipse type region, which depends on the Richardson number, wave number and depth of the fluid layer. For flows with $U^{*}_{min} \neq 0$, this region reduces to the line $c_i = 0$ when $J_o \to \frac{1}{4}$ - in accord with Miles' theorem (Chimonas - 1970). Under an approximation due to Shivamoggi (1977), the role of curvature on the stability is also studied.

CHAPTER 1

INTRODUCTION

1. The Stability Problem.

Consider the motion of an inviscid compressible fluid confined between two horizontal infinite rigid planes. The governing equations are the Euler equation

$$\rho \left(\frac{\partial \vec{U}}{\partial t} + (\vec{U}.\nabla)\vec{U}\right) = \rho\vec{g} - \nabla p , \tag{1}$$

the equation of continuity

$$\frac{\partial \rho}{\partial t} + (\vec{U}.\nabla)\rho + \rho\nabla.\vec{U} = 0, \tag{2}$$

and the equation of state

$$p = p(\rho, S) \tag{3}$$

For simplicity, we shall consider only isentropic flows for which

$$\frac{\partial S}{\partial t} + (\vec{U}.\nabla)S = 0, \tag{4}$$

where $\vec{U}$ is the velocity, ρ the density, p the pressure - S the entropy of the fluid and $\vec{g}$ is the acceleration due to gravity.

Equations (3), (4) and (2) can be combined to get

$$\frac{\partial p}{\partial t} + (\vec{U}.\nabla)p + a_*^2 \rho\nabla.\vec{U} = 0 \tag{5}$$

where $a_*^2 = \frac{\partial p}{\partial \rho})_S$ is the square of sound speed. The boundary conditions are that the vertical component of the velocity vanishes on the rigid walls at $y = y_1, y_2$.

If we fix a coordinate system with the x-axis along the horizontal direction and y-axis along the vertical direction, then we see that the flow variables given by

$$\vec{U} = U(y)\ i, \quad \rho = \rho(y) \text{ and } p = p(y), \tag{6}$$

where i is the unit vector along the x-axis, satisfy the governing equations (1), (2) and (5) and also the boundary conditions provided

$$\frac{dp}{dy} = -\rho g \tag{7}$$

Here, we allow $U(y)$, $\rho(y)$ and $p(y)$ to be any twice continuously differentiable functions. These fields define the basic flow. "Yet not every solution of the equations of motion, even if it is exact, can actually occur in Nature. The flows that occur in Nature must not only obey the equations of fluid dynamics, but also be STABLE" - Landau and Lifschitz (1959).

If the basic flow is disturbed slightly, the disturbance may either die away, persist as a disturbance of similar magnitude or grow so much that the basic flow becomes a different laminar or a turbulent flow. Broadly speaking, we call such disturbances (asymptotically) stable, neutrally stable or unstable respectively. All possible slight disturbances are

likely to be excited in some degree by small irregularities or vibrations of the basic flow in practice, so the basic flow will persist only if it is stable to all slight disturbances. Mathematically, we can define stability in the sense of Lyapunov, but in most applications common sense makes a formal definition unnecessary.

Now, we shall study the stability of the basic flow given by (6) and (7) to two-dimensional, in finitesimal, wavy disturbances. Such disturbances are called normal modes.

Let the disturbed motion be given by

$$\vec{U} = (U(y) + u'(x,y,t),\ v'(x,y,t))$$
$$\rho = \rho(y) + \rho'(x,y,t)\ ,\ p = p(y)+p'(x,y,t) \tag{8}$$

For infinitesimal disturbances the equations can be linearized and the linearized perturbation equations are

$$\rho\left[\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + v'\frac{dU}{dy}\right] = -\frac{\partial p'}{\partial x} \tag{9}$$

$$\rho\left[\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x}\right] = -\frac{\partial p'}{\partial y} - \rho' g \tag{10}$$

$$\frac{\partial \rho'}{\partial t} + U\frac{\partial \rho'}{\partial x} + v\frac{d\rho}{dy} + \rho\left[\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right] = 0 \tag{11}$$

$$\frac{\partial p'}{\partial t} + U\frac{\partial p'}{\partial x} - \rho g\, v' + \rho a_*^2\left[\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right] = 0 \tag{12}$$

For wavy disturbances we can take the normal modes

$$
\begin{aligned}
u' &= \hat{u}(y) \exp\left[ik\,(x - ct)\right] \\
v' &= \hat{v}(y) \exp\left[ik\,(x - ct)\right] \\
\rho' &= \hat{\rho}(y) \exp\left[ik\,(x - ct)\right] \\
p' &= \hat{p}(y) \exp\left[ik\,(x - ct)\right]
\end{aligned} \tag{13}
$$

and equations (9) - (12) lead to the equations

$$\rho\left[ik\,(U - c)\hat{u} + U'\hat{v}\right] = -\,ik\hat{p} \tag{14}$$

$$\rho\left[ik\,(U - c)\hat{v}\right] = -\,\hat{p}' - \hat{\rho} g \tag{15}$$

$$ik\,(U - c)\hat{\rho} + \rho'\,\hat{v} + \rho\left[ik\,\hat{u} + \hat{v}'\right] = 0 \tag{16}$$

$$ik\,(U - c)\,\hat{p} - \rho g\,\hat{v} + \rho a_*^2\left[ik\,\hat{u} + \hat{v}'\right] = 0 \tag{17}$$

In the four equations and hearafter, primes denote differentiat[ion] with respect to y. Elimination of all disturbance variables except v from (14) - (17) gives

$$
\begin{aligned}
\rho k^2(U-c)\,\hat{v} = {} & \left[\frac{\rho(U-c)\,\hat{v}' - \frac{\rho g}{a_*^2}(U-c)\hat{v} - \rho U'\,\hat{v}}{1 - \frac{(U-c)^2}{a_*^2}}\right]' \\
& + \frac{g}{a_*^2}\left[\frac{\rho(U-c)\,\hat{v}' - \frac{\rho g}{a_*^2}(U-c)\hat{v} - \rho U'\,\hat{v}}{1 - \frac{(U-c)^2}{a_*^2}}\right] \\
& + \frac{\rho N^2\,\hat{v}}{(U-c)}
\end{aligned} \tag{18}
$$

Here $\quad N^2 = -g\left(\frac{\rho'}{\rho} + \frac{g}{a_*^2}\right)$ $\qquad$ (19)

is the Brunt-Väissälä frequency.

Equation (18) can be rewritten as

$$\left[\frac{r(U-c)H' - rU'H}{1 - \frac{(U-c)^2}{a_*^2}}\right]' - rk^2\,(U-c)\,H + \frac{rN^2H}{(U-c)} = 0 \qquad (20)$$

$$\text{where } H(y) = v(y)\,\exp\left(-\int^{y} \frac{g}{a_*^2}\,dy'\right)$$

$$(21)$$

$$\text{and} \quad r(y) = (y)\,\exp\left(\int^{y} \frac{2g}{a_*^2}\,dy'\right)$$

The associated boundary conditions are

$$H(y_1) = 0 = H(y_2) \qquad (22)$$

The phase velocity $c = (c_r + ic_i)$ may be complex and the wave number k is taken to be positive. It is assumed that $N^2(y) \geq 0$ which means that the stratification is statically stable.

The stability equation (20) and equation (22) define an eigenvalue problem for the phase velocity c. If all the eigenvalues are real, then the basic flow is stable to two-dimensional, infinitesimal, wavy disturbances. The flow is unstable if there is a complex eigenvalue $c = (c_r + ic_i)$ with $c_i > 0$.

Let H be an unstable solution. Then the transformation $H = (U-c)F$ transforms (20) to

$$\left[\frac{r(U-c)^2 F'}{1-\frac{(U-c)^2}{a_*^2}}\right]' - rk^2 (U-c)^2 F + rN^2F = 0 \tag{23}$$

and the associated boundary conditions are

$$F(y_1) = 0 = F(y_2) \tag{24}$$

Since $c_i > 0$ for an unstable mode, a proper meaning can be attached to the transformation

$$G = (U-c)^{\frac{1}{2}}F \tag{25}$$

by taking a definite branch of it. The transformed equation in terms of G is

$$\left[\frac{r(U-c)G'}{1-\frac{(U-c)^2}{a_*^2}}\right]' - \frac{1}{2}\left[\frac{rU'}{1-\frac{(U-c)^2}{a_*^2}}\right]' G - \frac{rU'^2 G}{4(U-c)(1-\frac{(U-c)^2}{a_*^2})}$$

$$- rk^2(U-c)G + \frac{rN^2G}{(U-c)} = 0 \tag{26}$$

and the associated boundary conditions are

$$G(y_1) = 0 = G(y_2) \tag{27}$$

2. <u>Homogeneous Shear Flows</u>

For incompressible constant density fluids the stability equation is the Rayleigh equation

$$H'' - k^2H - \frac{U''}{U-c} H = 0 \tag{28}$$

and the boundary conditions are the same as (22). In 1880, Rayleigh proved his famous inflexion point theorem namely, a necessary condition for instability is that U'' vanishes atleast once in the flow domain. This result was extended by Fjortoft (1950) and Hoiland (1951). None of these conditions are sufficient conditions, as is evident from the counterexample of Tollmien (1935). However, Tollmien (1935) and Lin (1955) have shown that these conditions are sufficient for the instability of symmetric profiles. Howard (1961) has proved the semicircle theorem by which the complex wave velocity for any unstable mode must lie inside the semicircle in the upper half plane with the range of basic velocity as diameter. Hoiland (1951) has also found an estimate for the growth rate of an unstable. Sattinger (1967) has found an estimate for the growth rate, which depends on the curvature of the basic flow profile and depth of the fluid layer. In the stability of homogeneous shear flows, the curvature of the basic velocity profile and the depth of the fluid layer play important roles. For a thorough discussion of this problem, one may be referred to Lin (1955), Drazin and Howard (1966) and Drazin and Reid (1981).

3. Stratified Incompressible Shear Flows

The stability of shear flows of a continuously stratified fluid is a fascinating phenomenon of great importance to meteorology. Helmholtz (1868) was the first to study the instability of an interface dividing two layers of inviscid

and incompressible fluids of different densities and in relative horizontal motion. However, the detailed and exhaustive study of this problem was done by Kelvin (1871) in context with the study of the generation of water waves. The stability equation for this problem is

$$(\rho H')' - \rho k^2 H - \frac{(\rho U')'}{U-c} H + \frac{\rho N^2 H}{(U-c)^2} = 0 \qquad (29)$$

which is obtained from (20) by taking $\frac{1}{a_*^2} = 0$ in it. The above equation is known as the Taylor-Goldstein equation in honour of their derivations and exploitations in 1931.

The first paper with general velocity and density distributions was published by Synge (1933). He generalized Rayleigh's theorem and obtained bounds for the growth rate which were later obtained independently by Yih (1957) and Drazin (1958). Miles (1961) proved a conjecture of Taylor namely the flow is stable if the Richardson number $J \geq \frac{1}{4}$ everywhere in the flow domain for a class of flows. Howard (1961) gave a simple proof of Miles' theorem and removed the restrictions on the velocity and density profiles. He also proved that for any unstable mode, the complex wave velocity must be inside the semicircle in the upperhalf plane, which has the range of basic velocity as diameter. Kochar and Jain (1979) improved upon this result by showing that the complex wave velocity for any unstable mode lies inside a semiellipse rather than a semicircle whose major diameter coincides with

Howard's semicircle, while minor diameter depends on stratification. This result was further generalized by Jain and Kochar (1983) and Makov and Stepanyants (1984) incorporating the wave number and the depth of the fluid layer. Drazin (1958) found an example which is stable when $J \geq 4$ and unstable when $J < \frac{1}{4}$.

Howard (1963) generalized the method of perturbing the stability boundary and finding the adjacent unstable modes, to the case of stratified flow. Huppert(1973) considered number of examples and established the validity of Howard's formula. Engevik (1973, 75, 78) gave a simple derivation of Howard's formula and derived the differential equation to find the stability boundary. Engevik et al. (1985) have discussed the perturbation of the stability even when Howard's formula is not applicable.

After the early work of Synge (1933), many workers have discussed the role of curvature of the basic velocity profile on the stability of stratified shear flows. Thorpe (1969) found a counterexample for Fjortoft's theorem. Yih (1974) gave his sufficient condition for stability, which is the strongest known generalization of Rayleigh's theorem. Banerjee et als. (1972, 74, 78) obtained instability conditions depending on the curvature of the basic velocity profile for certain class of profiles.

Banks et al. (1976) examined the overall pattern of normal modes. For a given flow and wave number the modes are divided into five classes some of which may be empty.

4. Compressible Shear Flows

Stability of parallel shear flow of a compressible fluid is of interest in meteorology. Landau (1944), Hatanaka (1947) and Miles (1958) studied the instability of a vortex sheet in a gas with respect to infinitesimal disturbances neglecting all diffusive processes. But Blumen et al. (1975) have casted doubt on the physical value of models which incorporate a vortex sheet. Unfortunately, however, there is not much systematic work on the stability characteristics of inviscid compressible parallel shear flows (Betchov and Criminale, 1967, Dandapat and Gupta, 1977).

Some general stability characteristics of gasdynamic shear flows, without including the external force field of gravity, have been obtained by Lees and Lin (1946). According to them, the disturbances can be classified as subsonic, sonic or supersonic depending on whether the relative phase speed $|U-c|$ is less than, equal to or greater than the sound speed, i.e. $|U-c| \lesseqgtr a_*$. The subsonic disturbances are the counterparts of the so-called inertial modes, which are solutions of the corresponding equations for incompressible fluids. The physical significance of the sonic disturbances is apparently not clear. Finally, supersonic disturbances correspond to

compression or sound waves. Eckart (1963) generalized Howard's semicircle theorem for gasdynamic shear flows. Chimonas (1970) extended Miles' theorem and Howard's estimate for the growth rate of unstable modes to gasdynamic shear flows. The extension of the Rayleigh stability criterion and Howard's semicircle theorem to compressible shear flows, obtained by Lees and Lin (1946) and Eckart (1963) respectively, have each been rederived by Blumen (1970) by a different approach. He also found a subsonic neutral solution of the stability equation when the basic flow is represented by the hyperbolic-tangent velocity profile and discussed unstable modes numerically. Shivamoggi (1977) studied this problem under the approximation $\frac{1}{a_*} \ll 1$ and $c_i \ll 1$. The works of Chimonas and Blumen have been extended to Spiral flows by Howard (1973) and Dandapat and Gupta (1975).

5. General Aspects

Linear stability analysis can indicate only instantaneous tendency of a laminar flow for small perturbations. The moment the flow is unstable and shows a tendency to grow the linear analysis will no more be valid. Sizable perturbations and instabilities subsequent to primary instability become important. But linearized analysis must be considered as the first step in any stability theory and it is a natural starting point for the description and definition of nonlinear problems. An identical relationship exists also

between stability of inviscid flows and viscous flows, in which the inviscid analysis is considered as the first step in any stability theory and natural starting point for the description and definition of viscous problems. For a discussion of these points one may be referred to Lin (1955), Chandrasekhar (1961). Eckhaus (1965), Joseph (1976) and Drazin and Reid (1981). Further, the instability mechanism is related to the over-reflexion of waves and this has been discussed in Acheson (1976) and Pellacani (1983).

6. Plan and Contributions of the Thesis.

The present work is devoted to the investigation of the temporal stability of inviscid parallel shear flows using the normal mode technique.

In the second chapter of this thesis we consider the reduction of Howard's semicircle bound on the range of the complex wave velocity of an arbitrary unstable mode in the stability problem of homogeneous shear flows. As pointed out earlier, Howard (1961) has shown that for instability of a homogeneous shear flow the complex wave velocity c of any arbitrary unstable mode must lie inside a semicircle in the upper half plane, with the range of the basic velocity as diameter. This elegant result does not contain any parameter that characterizes the curvature of the basic profile or the depth of the fluid layer. The famous inflexion point theorem of Rayleigh (1880) and its subsequent extensions

by Holand (1951) and Fjortoft (1950) show the important role played by the curvature of the basic velocity profile on the stability of homogeneous shear flows. The estimates for growth rate of an unstable mode of Sattinger (1967) and Craik (1972) and also the example of Tollmien (1935) show the role of the depth of the fluid layer on the stability of homogeneous shear flows. In the present work Howard's semicircle bound is further reduced. The reduction depends on the curvature of the basic velocity profile and also on the depth of the fluid layer.

If, instead of a homogeneous shear flow, one considers a hetrogeneous shear flow, then gravity plays a dominant role on stability characteristics. For this problem, the role of curvature of the basic velocity profile on the stability characteristics is not clear. In fact the validity of the results available for homogeneous shear flows, in the presence of stratification have not been established so far. This motivates us to study, in the third chapter, the role of curvature of the basic velocity profile on the stability of stratified shear flows.

First, we give a simple generalization of Hoiland's result to the stratified case. Then, we generalize Sattinger's estimate for the growth rate of an unstable mode to the stratified case. This estimate involves the curvature and we note that its improvement by Craik (1972) cannot be carried over to

the stratified case. Recently Kochar and Jain (1979) have proved that the unstable modes of a statically stratified shear flow lies inside a certain semiellipse in the upper half-plane, whose major axis coincides with the diameter of Howard's semicircle, while its minor axis depends on the stratification parameter viz. the Richardson number. In our work, we show that the unstable modes lie also inside a semiellipse region whose major axis coincides with the diameter of Howard's semicircle, while its minor axis depends not only on the stratification but also on the curvature of the basic velocity profile. Lastly, we extend the results of the previous chapter to the stratified case. It is interesting to note that these results are also improvements over the results of Banerjee et al. (1974, 78).

In the fourth chapter, we discuss the boundedness of the phase velocities of the normal modes of the Taylor-Goldstein problem. The works of Howard (1961) and Kochar and Jain (1979) show that the complex wave velocity of unstable and their conjugate damped modes lie in a bounded region in the $c_r c_i$-plane. The wave velocity of singular neutral modes, by their very definition, lie between the minimum and maximum values of the basic velocity. But, what about the boundedness of the phase velocities of non-singular neutral modes? We show that the wave velocity of an arbitrary non-singular neutral mode of the Taylor-Goldstein problem is also bounded. We give explicit bounds.

In the fifth chapter, we study the stability of compressible shear flows without the presence of gravity. We show that the basic flow $U \equiv 0$ supports supersonic waves but not subsonic waves. The semicircle theorem has been proved for compressible shear flows by Eckart (1962) and later by Blumen (1970). But this result does not incorporate the effect of compressibility and remains the same as that for incompressible fluids. We show that the complex wave velocities of an unstable subsonic disturbances lie inside a semiellipse type region depending on the Mach number. The semicircle comes out to be a special case of this when $M = 0$. Incidently, this region comes out to be depth as well as wave number dependant. When the flow domain is unbounded, we have rederived Hoiland's estimate for the growth rate of any unstable mode by a different method and have also found a sufficient condition for stability to supersonic disturbances. We show by an example that the phase velocities of non-singular neutral modes of compressible flows are not always bounded.

In the sixth and final chapter, we study the stability of parallel compressible shear flows in the presence of gravity. We modify Chimonas' (1970) proof of Miles' theorem. This enables us to show that the instability region for subsonic disturbances is not Eckart's semicircle, but rather a semiellipse type region which depends on the Richardson number and also on the depth of the fluid layer and wave number.

Furthermore, when $U'_{min} \neq 0$, this region reduces to the line $c_i = 0$ when $J_o \to \frac{1}{4}$ - as expected from Miles' theorem. Following Shivamoggi (1977), we take $\frac{1}{a_*^2}$ and the imaginary part of complex wave velocity c_i to be very small compared to unity so that their product can be neglected in comparison to unity. Under this approximation, we study the role of curvature of the basic velocity profile on the stability of the flows. For subsonic disturbances, we have generalized many standard results known for incompressible flows like instability criterion and estimate for the growth rate, due to Synge.

CHAPTER - 2

ON REDUCING HOWARD'S SEMICIRCLE FOR HOMOGENEOUS SHEAR FLOWS

1. Introduction

In the stability problem of homogeneous shear flows Howard (1961) established that the complex wave velocity $c = (c_r + ic_i)$ of an arbitrary unstable mode $(c_i > 0)$ must be inside or on the semicircle

$$\left(c_r - \frac{a+b}{2}\right)^2 + c_i^2 = \left(\frac{b-a}{2}\right)^2 \tag{1}$$

where $a = U_{min}$ and $b = U_{max}$, U being the basic velocity profile. This elegant result does not appear to have been improved in the literature. The famous inflexion point theorem of Rayleigh and its subsequent extensions by Fjortoft (1950) and Hoiland (1951) show the important role played by the curvature of the basic velocity profile on the stability or instability of homogeneous shear flows. However, Howard's semicircle theorem does not contain any parameter that characterizes the curvature of the basic velocity profile. In this chapter, it is shown that Howard's semicircle bound on the complex wave velocity of an arbitrary unstable mode can further be reduced. The reduction depends on the curvature of the basic velocity profile and also on the depth of the fluid layer. The role of the depth of the fluid layer on the stability of homogeneous

shear flows have been brought out in the works of Sattinger (1967), Craik (1972) and Tollmien (1935).

2. Mathematical Analysis

The basic equation and the boundary conditions for the problem of homogeneous shear flows are given by (cf. Chapter 1)

$$[(U-c)G']' - [\frac{U''}{2} + k^2(U-c) + \frac{U'^2}{4(U-c)}]\, G = 0, \tag{2}$$

$$G(y_1) = 0 = G(y_2) \tag{3}$$

where the primes denote differentiation w.r.t. y, k is the wave number and $c = c_r + ic_i$ is the complex wave velocity.

Theorem 1.

If (c,G), $c = c_r + ic_i$, $c_i > 0$ is a solution of equations (2) - (3) and $f = U'' + \frac{2b\,\pi^2}{(y_1-y_1)^2} > 0$, for every $y \varepsilon [y_1, y_2]$, then

$$c_i^2 \leq \lambda\,(c_r - \frac{a}{m+1}) \tag{4}$$

where $\lambda = [\frac{U'^2(m+1)}{2f}]_{max}$ and $m = \frac{b}{a}$, $a > 0$.

Proof.

Multiplying (2) by G^* (the complex conjugate of G) and integrating over the range of y, we have upon integration by parts once and making use of (3),

$$\int (U-c) [|G'|^2+k^2|G|^2] + \frac{1}{2} \int U'' |G|^2 dy + \int \frac{U'^2|G|^2}{4(U-c)} = 0 \tag{5}$$

The limits of integration and the infinitesimal length dy is dropped throughout for convenience. Equating the real and imaginary parts of (5) to zero and cancelling c_i (> 0), we get

$$\int (U-c_r) [|G'|^2 + k^2|G|^2] + \frac{1}{2} \int U'' |G|^2 + \frac{1}{4} \int \frac{U'^2(U-c_r) |G|^2}{[(U-c_r)^2 + c_i^2]} = 0 \tag{6}$$

$$\int [|G'|^2 + k^2|G|^2] - \frac{1}{4} \int \frac{U'^2 |G|^2}{[(U-c_r)^2 + c_i^2]} = 0. \tag{7}$$

Multiplying (7) by mc_r and adding the resulting equation to (6), we get

$$\int [U+(m-1)c_r] [|G'|^2 + k^2|G|^2] + \frac{1}{2} \int U'' |G|^2 + \frac{1}{4} \int \frac{U'^2 [U - (m+1)c_r]|G|^2}{[(U-c_r)^2 + c_i^2]} = 0 \tag{8}$$

Since $a > 0$, and $a < c_r < b$ (Rayleigh, 1880), it follows that

$$[U+(m-1)c_r]_{min} = b > 0 \text{ and } [U-(m+1)c_r]_{max} = -a < 0 \tag{9}$$

Equation (8) upon using (9) and the Rayleigh-Ritz inequality (Shultz, 1973)

$$\int |G'|^2 \geq \frac{\pi^2}{(y_2-y_1)^2} \int |G|^2, \tag{10}$$

gives

$$\int \frac{[2c_i^2 f - U'^2 \{(m+1)c_r - U\}]\, |G|^2}{4\,[(U-c_r)^2+c_i^2]} \leq 0 \tag{11}$$

Under the conditions of the theorem, (11) clearly implies that

$$c_i^2 \leq \lambda\,(c_r - \frac{a}{m+1}),$$

where $\lambda = [\frac{U'^2\,(m+1)}{2f}]_{max}$,

and this establishes the theorem.

Theorem 2.

Under the conditions of Theorem 1, if

$$\lambda < (\frac{m-1}{m+1}\,a+b) - [(\frac{m-1}{m+1}\,a+b)^2 - (b-a)^2]^{1/2}, \tag{12}$$

then the parabola

$$c_i^2 = \lambda\,(c_r - \frac{a}{m+1}), \tag{13}$$

intersects Howard's semicircle (1).

Proof.

It is easily seen that the parabola (13) touches Howard's semicircle (1), if

$$\lambda = \lambda_c = (\frac{m-1}{m+1}\,a+b) \pm [(\frac{m-1}{m+1}\,a+b) - (b-a)^2]^{1/2} \tag{14}$$

The value of λ_c given by (14) with the positive sign is rejected as it leads to $c_r < a$ which violates $a < c_r < b$. Hence, if

$$\lambda < \left(\frac{m-1}{m+1}\, a+b\right) - \left[\left(\frac{m-1}{m+1}\, a+b\right)^2 - (b-a)^2\right]^{1/2}$$

then the parabola (13) intersects Howard's semicircle (1).

This proves the theorem.

Theorem 3.

If (c,G), $c = c_r + ic_i$, $c_i > 0$ is a solution of equations (2) - (3) and every $y \in [y_1, y_2]$ then

$$c_i^2 \leq \lambda^* \left(c_r + \frac{b}{m-1}\right), \tag{15}$$

where $\lambda^* = \left[\frac{U'^2(m-1)}{2|g|}\right]_{max}$ and $m = \frac{b}{a}$, $a > 0$.

Proof. Multiplying (7) by $-mc_r$, adding the resulting equation to (6), and proceeding as in Theorem 1, we get the result.

Theorem 4.

Under the conditions of Theorem 3, if

$$\lambda^* \leq \left(\frac{m+1}{m-1}\, b+a\right) - \left[\left(\frac{m+1}{m-1}\, b+a\right)^2 - (b-a)^2\right]^{1/2}, \tag{16}$$

then the parabola

$$c_i^2 = \lambda^* \left(c_r + \frac{b}{m-1}\right), \tag{17}$$

intersects Howard's semicircle (1).

Proof. Follows by proceeding as in Theorem 2.

3. Concluding Remarks

Theorems 1 - 4 in conjunction with Howard's semicircle clearly show that if

$$a > 0 \text{ and either } f > 0 \text{ or } g < 0, \tag{18}$$

for every $y \varepsilon [y_1, y_2]$, then under the conditions (12) or (16), Howard's semicircle bound for homogeneous shear flows can further be reduced. Further since the conditions (18) allow the basic velocity profile to change the sign of its curvature somewhere in the flow domain the result assumes more significance because under these conditions the basic velocity profile if symmetric is unstable (Tollmien, 1935).

CHAPTER - 3

ON THE ROLE OF CURVATURE IN THE STABILITY OF HETROGENEOUS SHEAR FLOWS

1. Introduction.

For a steady, plane, parallel flow of an inviscid, incompressible, homogeneous fluid confined between two parallel rigid planes, a necessary condition for instability is that the basic velocity profile has atleast one inflexion point in the flow domain. This is the famous Rayleigh's theorem. Hoiland (1951) extended this result further by proving that a necessary condition for instability is that $U''(U-c_r) < 0$ atleast once in the flow domain, where $U(y)$ is the basic velocity and $c = c_r + ic_i$, $c_i > 0$ is the phase velocity of an unstable normal mode. Fjortoft (1950) gave a stronger result by showing that a necessary condition for instability is $U''(U - U_s) < 0$ atleast once in the flow domain where $U_s = U(y_s)$ such that $U''(y_s) = 0$. For a discussion of these results and their physical interpretation, one may be referred to Drazin and Reid (1981). These results illustrate the important role played by the curvature of the basic velocity in the stability of homogeneous shear flows.

However, instead of a homogeneous fluid, if one considers a horizontally stratified fluid under the action of gravity; then the question : What is the role of curvature

on the stability of the flow? does not have a satisfactory answer till today. In fact, the validity of the Rayleigh's result in the presence of stratification has not been established so far.

As early as 1933, Synge gave a generalization of Rayleigh's theorem. Later on, Yih (1974) gave his sufficient condition for stability, which is the strongest known generalization of Rayleigh's theorem. Unfortunately, both these results are not as simple as the Rayleigh's theorem. Trorpe (1969) has found an example of a basic flow which is stable in the homogeneous case by Fjortoft's theorem, but which is unstable in the stratified case. In this chapter, we give a simple generalization of Hoiland's result to the stratified case.

Howard (1961) was the first to give an estimate for the growth rate of an unstable mode of a stratified fluid. For homogeneous fluid, his estimate reduces to Hoiland's estimate of the growth rate. Recently, Makov and Stepanyants (1984) gave a generalization of Howard's estimate taking into account the total depth of the fluid layer with velocity shear. All these estimates do not take into account the curvature effects on the growth rate.

For a homogeneous fluid, Sattinger (1967) gave an estimate of the growth rate of an unstable disturbance involving the curvature of the basic flow. This has been improved by

Craik (1972). In this chapter, Sattinger's estimate is generalized to the stratified case as it is not possible to do the same for Craik's estimate.

Recently Kochar and Jain (1979) proved that the complex wave velocity of unstable modes of a statically stratified fluid lies inside a certain semi-ellipse in the upper half-plane, whose major axis coincides with the diameter of Howard's semicircle, while its minor axis depends on the stratification parameter viz. the Richardson number. We show that the unstable modes lie also inside a semiellipse region whose major axis coincides with the diameter of Howard's semicircle, while its minor axis depends not only on the stratification but also on the curvature of the basic velocity profile.

Lastly, we extend the results of the previous chapter to the stratified case. We derive a necessary condition for instability which simultaneously gives Miles' criterion, a range of c_r and c_i and also takes into account the curvature effects on the stability of flows. It is interesting to note that these results are improvements over the results of Banerjee et al. (1974, 1978).

2. Hoiland's Criterion.

The Taylor-Goldstein system is

$$(\rho H')' - \rho k^2 H - \frac{(\rho U')'}{U-c} H + \frac{\rho N^2 H}{(U-c)^2} = 0 \qquad (1)$$

with the associated boundary conditions

$$H(y) = 0 \text{ at } y = y_1, y_2, \ldots . \tag{2}$$

If (1) is multiplied by H^* (complex conjugate of H) and integrated over (y_1, y_2), we get, after an integration by parts

$$\int \rho(|H'|^2 + k^2|H|^2) + \int \frac{(\rho U')'}{U-c} |H|^2 - \int \frac{\rho N^2 |H|^2}{(U-c)^2} = 0 \tag{3}$$

The limits of integration and the infinitesimal length dy are dropped throughout for convenience. The real part of (3) is

$$\int \rho(|H'|^2 + k^2|H|^2) + \int \frac{(\rho U')'(U-c_r)}{|U-c|^2} |H|^2 - \int \frac{\rho N^2 [(U-c_r)^2 - c_i^2]|H|^2}{|U-c|^4} = 0 \tag{4}$$

Using the fact $(U-c_r)^2 - c_i^2 = |U-c|^2 - 2c_i^2$, (4) can be rewritten as

$$\int \frac{(\rho U')'(U-c_r) - \rho N^2}{|U-c|^2} |H|^2 = -\int \rho(|H'|^2 + k^2|H|^2) - 2c_i^2 \int \frac{\rho N^2 |H|^2}{|U-c|^4} \tag{5}$$

The right hand side of (5) is negative. Therefore, a necessary condition for instability is that

$$(\rho U')' \ (U-c_r) < \rho N^2 \tag{6}$$

atleast at one point in (y_1, y_2). Since for a homogeneous fluid ρ = constant, $N^2 = 0$, the condition becomes

$$U'' \ (U - c_r) < 0 \tag{7}$$

atleast at one point in the flow domain. This is Hoiland's criterion for the Rayleigh problem. Thorpe (1969) has shown that a stable flow of a homogeneous fluid with velocity profile $U(y) = c - \sinh y$ becomes unstable, if the density profile is given by $N^2 = J_o + J_1 \tanh^2 y$, $J_o > 0$. It is easy to see that these profiles do not violate condition (6).

3. <u>An estimate for the growth rate.</u>

From (4), we have

$$\int \rho(|H'|^2 + k^2|H|^2) = - \int \frac{(\rho U')' \, (U-c_r)}{|U-c|^2} |H|^2$$

$$+ \int \frac{\rho N^2 \left[(U-c_r)^2 - c_i^2 \right]}{|U-c|^4} |H|^2 \qquad (8)$$

Taking into account that

$$- (\rho U')' \, (U-c_r) \leq |(\rho U')'| \, |U-c_r|$$

$$\leq |(\rho U')'|_{max} \, (U_{max} - U_{min})$$

and $(U-c_r)^2 - c_i^2 \leq |U-c|^2$

we obtain from (8)

$$\int \rho(|H'|^2 + k^2|H|^2) \leq |(\rho U')'|_{max} \, (U_{max} - U_{min}) \int \frac{|H|^2}{|U-c|^2}$$

$$+ (\rho N^2)_{max} \int \frac{|H|^2}{|U-c|^2} \qquad (9)$$

Using the well known Poincare inequality, we have

$$\int \rho(|H'|^2 + k^2|H|^2) \geq \rho_{min} \left(\frac{\pi^2}{(y_2-y_1)^2} + k^2\right) \int |H|^2 \qquad (10)$$

Using (10) and the fact $\frac{1}{|U-c|^2} \leq \frac{1}{c_i^2}$ in (9), we get the following estimate :

$$k^2 c_i^2 \leq \frac{|(\rho U')'|_{max} (U_{max} - U_{min}) + (\rho N^2)_{max}}{\min \left(1 + \frac{\pi^2}{k^2(y_2-y_1)^2}\right)} \qquad (11)$$

For a homogeneous fluid (ρ = constant), this reduces to Sattinger's estimate

$$k^2 c_i^2 \leq \frac{|U''|_{max} (U_{max} - U_{min})}{\left(1 + \frac{\pi^2}{k^2(y_2-y_1)^2}\right)} \qquad (12)$$

From (12) we see that constant vorticity flows of a homogeneous fluid are stable. But we can not make such a conclusion from (11). Therefore constant vorticity flows, which are stable when the fluid is homogeneous, may become unstable when the fluid is stratified.

4. A Semiellipse Theorem.

If we introduce the transformation $H = (U-c)F$, then the Taylor-Goldstein system becomes

$$[\rho(U-c)^2 F']' + \rho[N^2 - k^2(U-c)^2] F = 0 \qquad (13)$$

and $\quad F(y_1) = 0 = F(y_2) \qquad (14)$

If the equation (13) is multiplied by F^* (complex conjugate of F) and integrated over (y_1, y_2), then the real and imaginary parts together with certain ingenius manipulations introduced by Howard (1961) imply the inequality

$$\left[(c_r - \frac{a+b}{2})^2 + c_i^2 - (\frac{b-a}{2})^2\right] \int \rho Q + \int \rho N^2 |F|^2 \leq 0 \qquad (15)$$

where $Q = |F'|^2 + k^2|F|^2$, $a \leq U(y) \leq b$. (16)

Howard drops the last term in (15) which is positive to establish the semicircle theorem. Kochar and Jain (1979) found a positive, lower and relevant estimate of the dropped term to get their semiellipse theorem. However, neither Howard's theorem, nor its generalization by Kochar and Jain take into account the dependence of the size of the instability region on the curvature of the basic flow. Now, we find a new estimate for the last term in (15) to get a new semi-ellipse theorem incorporating the curvature effect

$$H = (U-c)F.$$

Differentiating this, we get

$$|H'| \geq \left| \, |U-c| \, |F'| - |U'| \, |F| \, \right|$$

Therefore, $\int \rho |H'|^2 \geq \int \rho |U-c|^2 |F'|^2 + \int \rho |U'|^2 |F|^2$

$$- 2 \int \rho \, |U-c| \, |U'| \, |F| \, |F'| \qquad (17)$$

Let $B^2 = \int \rho U'^2 |F|^2$; $E^2 = \int \rho |U-c|^2 Q$, $J_o = [\frac{N^2}{U'^2}]_{min}$,

where J(y) denotes the local Richardson number.

By Cauchy-Schwarz inequality

$$\int \rho|U-c|\;|F|\;|F'| \leq [\int \rho U'^2|F|^2 \int \rho|U-c|^2|F'|^2]^{1/2}$$

$$\leq BE \tag{18}$$

Addition of $\int \rho k^2|H|^2$ to both sides of (17) and use of (18) gives

$$\int \rho(|H'|^2 + k^2|H|^2) \geq (B-E)^2 \tag{19}$$

Using (9), we get

$$(B-E)^2 \leq [\,|(\rho U')'|_{max}\,(b-a) + (\rho N^2)_{max}] \int |F|^2 \tag{20}$$

Now, since $\int |F|^2 \leq \int \dfrac{\rho U'^2\,|F|^2}{\rho_{min}\,U'^2_{min}}$, provided $U'_{min} \neq 0$,

we obtain from (20),

$$\left(\frac{E}{B} - 1\right)^2 B^2 \leq \frac{[\,|(\rho U')'|_{max}\,(b-a) + (\rho N^2)_{max}\,]B^2}{\rho_{min}\,U'^2_{min}}$$

$$\therefore\quad \frac{E}{B} \leq 1 + A \tag{21}$$

where $A^2 = \dfrac{|(\rho U')'|_{max}\,(b-a) + (\rho N^2)_{max}}{\rho_{min}\,U'^2_{min}}$, $U'_{min} \neq 0$.

Therefore, $\int \rho N^2\,|F|^2 \geq J_o\,B^2$

$$\geq \frac{J_o\,E^2}{(1+A)^2}$$

$$\geq \frac{J_o\, c_i^2}{(1+A)^2} \int \rho Q.$$

Thus we have established the following lemma.

Lemma 1.

For an unstable mode, we have

$$\int \rho N^2 |F|^2 \geq \frac{J_o\, c_i^2 \int \rho Q}{\left\{1 + \left[\frac{|(\rho U')'|_{max}(b-a) + (\rho N^2)_{max}}{\rho_{min}\, U'^2_{min}}\right]^{1/2}\right\}^2} \qquad (22)$$

provided $U'_{min} \neq 0$.

Let us use the estimate given by the lemma of the last term in the inequality (15) to get

$$\left\{(c_r - \frac{a+b}{2})^2 + c_i^2\left[1 + \frac{J_o}{\left\{1 + \left[\frac{|(\rho U')'|_{max}(b-a)+(\rho N^2)_{max}}{\rho_{min}\, U'^2_{min}}\right]^{1/2}\right\}^2}\right] - (\frac{b-a}{2})^2\right\} \int \rho Q \leq 0.$$

This implies the following theorem.

Theorem 1.

For flows U with $U'_{min} \neq 0$, the complex wave velocity c for an unstable mode must lie inside the semiellipse

$$(c_r - \frac{a+b}{2})^2 + \left[1 + \frac{J_o}{\left\{1 + \left[\frac{|(\rho U')'|_{max}(b-a) + (\rho N^2)_{max}}{\rho_{min}\, U'^2_{min}}\right]^{1/2}\right\}^2}\right] c_i^2 \leq (\frac{b-a}{2})^2 \qquad (23)$$

In particular, for a homogeneous fluid for which $J_0 = 0$, this semiellipse theorem reduces to Howard's semicircle theorem. Jain and Kochar (1983) and Makov and Stepanyants (1984) have given generalizations of the Kochar-Jain semiellipse theorem. Following them, similar generalizations can be given for our semiellipse theorem also. However, they do not seem to throw any additional light on the role of curvature on the stability of the flow and hence, they are not given here.

5. Reduction and Unification Problem.

Banerjee et al. (1974, 1978) have established a necessary condition for instability which simultaneously gives Miles' criterion and a range of c_r and c_i for an unstable mode for a class of velocity and density distributions satisfying (i) $a > 0$ and (ii) $(\rho U')' \leq 0$ or $(\rho U')' \geq 0$ everywhere in the flow domain. Gupta et al. (1982) have removed the restriction (i) and obtained the results. All these results have the inherent drawback that they can not be applied to homogeneous shear flows for in this case condition (ii) in view of Rayleigh's inflexion point theorem implies that the system is stable. Now, we shall relax the very restrictive condition (ii) and thereby generalize the results of the previous chapter to the stratified case.

The stability equation in terms of $G = (U-c)^{1/2} F$ is

$$[\rho(U-c)G']' - \left[\frac{(\rho U')'}{2} + \rho k^2(U-c) + \frac{\rho\left(\frac{U'^2}{4} - N^2\right)}{U-c}\right]G = 0 \tag{24}$$

and the associated boundary conditions are

$$G(y_1) = 0 = G(y_2) \qquad (25)$$

Theorem 2.

If (i) $a > 0$ and

$$\text{(ii)}\ f(y) = [(\rho U')' + \frac{2\,\rho_{min}\, b\pi^2}{(y_2-y_1)^2}] > 0,$$

for every $y \varepsilon [y_1, y_2]$,

then $c_i^2 \leq \lambda\,(c_r - \frac{a}{m+1})$, (26)

where $\lambda = \max\limits_{[y_1,y_2]} \{\frac{2\ \rho(m+1)\ (\frac{U'^2}{4} - N^2)}{f(y)}\}$ and $m = \frac{b}{a}$.

Proof.

Multiplying (24) by G^* (complex conjugate of G) and integrating over the range of y, we have upon integration by parts and making use of (25),

$$\int \rho\,(U-c)\,[|G'|^2 + k^2|G|^2] + \frac{1}{2}\int (\rho U')'\ |G|^2 + \int \frac{\rho(\frac{U'^2}{4} - N^2)}{(U-c)}\ |G|^2 = 0 \qquad (27)$$

Equating the real and imaginary parts of (27) to zero and cancelling c_i (> 0), we get

$$\int \rho\,(U-c_r)\,[|G'|^2 + k^2|G|^2] + \frac{1}{2}\int (\rho U')'\ |G|^2 + \int \frac{\rho(U-c_r)(\frac{U'^2}{4} - N^2)\,|G|^2}{(U-c_r)^2 + c_i^2} = 0, \qquad (28)$$

$$\int \rho \left[|G'|^2 + k^2 |G|^2 \right] + \int \frac{\rho (N^2 - \frac{U'^2}{4})}{(U-c_r)^2 + c_i^2} |G|^2 = 0 \qquad (29)$$

Multiplying (29) by mc_r and adding the resulting equation to (28), we get

$$\int \rho[U + (m-1)c_r] \, (|G'|^2 + k^2|G|^2) + \frac{1}{2} \int (\rho U')' \, |G|^2$$
$$+ \int \frac{\rho\,[U-(m+1)c_r] \, (\frac{U'^2}{4} - N^2) \, |G|^2}{(U-c_r)^2 + c_i^2} = 0 \qquad (30)$$

Since $a > 0$, and $a < c_r < b$ (Miles, 1961), it follows that

$$[U + (m-1)c_r]_{min} = b > 0 \text{ and } [U-(m+1)c_r]_{max} = -a < 0 \qquad (31)$$

Equation (30) upon using (31) and the Rayleigh-Ritz inequality gives

$$\int \frac{f(y)c_i^2 + 2\rho \, \{U-(m+1)c_r\} \, (\frac{U'^2}{4} - N^2) \quad |G|^2}{(U-c_r)^2 + c_i^2} \leq 0 \qquad (32)$$

Under the conditions of the theorem, (32) clearly implies that

$$c_i^2 \leq \lambda(c_r - \frac{a}{m+1}),$$

where $\lambda = \max\limits_{[y_1, y_2]} \left\{ \frac{2\rho \; (m+1) \; (\frac{U'^2}{4} - N^2)}{f(y)} \right\}$,

and this establishes the theorem.

Theorem 3.

Under the conditions of theorem 2, if

$$\lambda < (\frac{m-1}{m+1} a+b) - [(\frac{m-1}{m+1} a+b)^2 - (b-a)^2]^{1/2} , \qquad (33)$$

then the parabola

$$c_i^2 = \lambda(c_r - \frac{a}{m+1}) \tag{34}$$

intersects Howard's semicircle

$$(c_r - \frac{a+b}{2})^2 + c_i^2 = (\frac{b-a}{2})^2 \tag{35}$$

Proof.

It is easily seen that the parabola (34) touches Howard's semicircle (35), if

$$\lambda = \lambda_c = (\frac{m-1}{m+1}\ a+b) \pm [(\frac{m-1}{m+1}\ a+b)^2 - (b-a)^2]^{1/2} \tag{36}$$

The value of λ_c given by (36) with the positive sign is rejected as it leads to $c_r < a$ which violates $a < c_r < b$. Therefore

$$\lambda_c = (\frac{m-1}{m+1}\ a+b) - [(\frac{m-1}{m+1}\ a+b)^2 - (b-a)^2]^{1/2}.$$

Hence, if $\lambda < (\frac{m-1}{m+1}\ a+b) - [(\frac{m-1}{m+1}\ a+b)^2 - (b-a)^2]^{1/2}$,

then the parabola (34) certainly intersects Howard's semicircle (35).

This establishes the theorem.

Theorem 4.

If (i) $a > 0$ and

(ii) $g(y) = [(\rho U')' - \frac{2\ \rho_{max}\ a\pi^2}{(y_2-y_1)^2}] < 0$

for every y in $[y_1, y_2]$, then

$$c_i^2 \leq \lambda^* \left(c_r + \frac{b}{m-1}\right), \tag{37}$$

where $\lambda^* = \max_{[y_1, y_2]} \left\{ \dfrac{2\,\rho(m-1)\left(\frac{U'^2}{4} - N^2\right)}{|g(y)|} \right\}$

and $\quad m = \frac{b}{a} > 1.$

Proof.

Multiplying (29) by $-mc_r$, adding the resulting equation to (28), and proceeding as in Theorem 2, we get the result.

Theorem 5.

Under the conditions of Theorem 4, if

$$\lambda^* < \left(\frac{m+1}{m-1}\,b+a\right) - \left[\left(\frac{m+1}{m-1}\,b+a\right)^2 - (b-a)^2\right]^{1/2}, \tag{38}$$

then the parabola

$$c_i^2 = \lambda^* \left(c_r + \frac{b}{m-1}\right), \tag{39}$$

intersects Howard's semicircle (35).

Proof.

Follows by proceeding as in Theorem 3.

From Theorems (2) or (4), it follows that a necessary condition for instability is that $J = \frac{N^2}{U'^2} < \frac{1}{4}$ somewhere inside the flow domain (Miles, 1961). Further Theorems 2 - 5 in conjunction with Howard's semicircle theorem clearly show that if

$a > 0$ and either $f(y) > 0$ or $g(y) < 0$, (40)

for every y in $[y_1, y_2]$, then under conditions (33) or (38), Howard's semicircle bound for hetrogeneous shear flows can further be reduced. The importance of condition (40) lies in the fact that under this condition the basic velocity profile can change the sign of its curvature in the flow domain. Further, for homogeneous shear flows they reduce to the results of Chapter 2.

6. Concluding Remarks.

Curvature of the basic velocity profile plays an important role in the stability analysis of inviscid, incompressible plane parallel flows in the absence of stratification. It is through curvature that a condition which is necessary as well as sufficient for the existence of unstable modes is available. In the presence of stratification one wonders whether curvature of the basic velocity profile has the same role to play? The present study has been motivated by the above question and all the results of this chapter explicitly incorporate the curvature effect on the stability of stratified shear flows. However, one still does not know whether the simple Rayleigh's theorem is true or not in the presence of stratification.

CHAPTER - 4

ON THE BOUNDEDNESS OF THE WAVE VELOCITY OF NON-SINGULAR NEUTRAL MODES OF THE TAYLOR-GOLDSTEIN PROBLEM

1. Introduction

For a given flow and wave number, the normal modes may be divided into five classes, some of which may be empty (Drazin and Reid, 1981). The investigations of Howard (1961) and Kochar and Jain (1979) show that the complex wave velocity $c = c_r + ic_i$ of unstable ($c_i > 0$) and their conjugate damped ($c_i < 0$) modes of the Taylor-Goldstein problem lie in a bounded region in the $c_r c_i$-plane. The wave velocity of singular neutral modes by their very definition ($c_i = 0$, $U-c = 0$ for some y in $[y_1, y_2]$) lies between the minimum and maximum values a and b respectively of the streaming velocity U. However the definition of non-singular neutral modes ($c_i = 0$, $U-c \neq 0$ for any y in $[y_1, y_2]$) implies that their wave velocity is either bounded below by b or bounded above by a. Therefore the possibility of their unboundedness still exists. In this chapter we rule out this possibility by proving that the wave velocity of an arbitrary non-singular neutral mode of the Taylor-Goldstein problem is bounded. However, if the fluid is compressible, the wave velocity of an arbitrary non-singular neutral mode may be unbounded as shown in the next chapter.

2. Establishment of the Result.

For a non-singular mode, we have $c = c_r$ and $U-c_r \neq 0$ in $[y_1, y_2]$. For illustration by Rayleigh's (1883) example, we take the Taylor-Goldstein equation under the Boussinesq approximation. Therefore, the Taylor-Goldstein equation may be written as

$$H'' + \left[\frac{N^2}{(U-c_r)^2} - \frac{U''}{(U - c_r)} - k^2\right] H = 0 \tag{1}$$

and the associated boundary conditions are

$$H(y_1) = 0 = H(y_2) \tag{2}$$

For a real eigenvalue $c = c_r$, the eigenfunction may be taken as a real valued function. Therefore, multiplying equation (1) by the eigenfunction H and integrating the resulting equation with the help of (2), we have

$$\int \left[|H'|^2 + k^2|H|^2 + \frac{U''}{U-c_r}|H|^2 - \frac{N^2|H|^2}{(U-c_r)^2}\right] = 0 \tag{3}$$

Using the Rayleigh-Ritz inequality (Shultz,1973) in equation (3) and dropping the positive term involving the wave number from the resulting inequality, we have

$$\int \left[(U-c_r)^2\alpha^2 + U''(U-c_r) - N^2\right]\frac{|H|^2}{(U-c_r)^2} < 0 \tag{4}$$

where $$\alpha^2 = \frac{\pi^2}{(y_2-y_1)^2} \tag{5}$$

Inequality (4) implies that there exists $y_s \in [y_1, y_2]$ such that

$$f(c_r) = \alpha^2 c_r^2 - [2U(y_s)\alpha^2 + U''(y_s)]\, c_r$$

$$+ [\alpha^2 U^2(y_s) + U(y_s)U''(y_s) - N^2(y_s)] < 0 \qquad (6)$$

When $N^2 > 0$, the discriminant of $f(c_r) = 0$, namely, $D(y_s) = U''^2(y_s) + 4\alpha^2 N^2(y_s)$ is positive. Therefore, it follows from inequality (6) that

$$A < c_r < B \qquad (7)$$

where $$\mathbf{A} = U(y_s) + \frac{U''(y_s) - \sqrt{D(y_s)}}{2\alpha^2} \qquad (8)$$

and $$B = U(y_s) + \frac{U''(y_s) + \sqrt{D(y_s)}}{2\alpha^2} \qquad (9)$$

From these, we have the following theorem.

<u>Theorem 1</u>. The phase velocities $c = c_r$ of non-singular neutral modes of the Taylor-Goldstein problem are bounded and

$$\frac{|U''|_{min} - \sqrt{|U''|^2_{max} + 4\alpha^2 N^2_{max}}}{2\alpha^2} < c_r < b + \frac{|U''|_{max} + \sqrt{|U''|^2_{mac} + 4\alpha^2 N^2_{max}}}{2\alpha^2}$$

(10)

where $\alpha^2 = \dfrac{\pi^2}{(y_2 - y_1)^2}$.

It is interesting to note that the bounds givenby (10) do not depend upon the wave number k. However, if we do not drop the term multiplied by k^2 to arrive at the equation (4), then we get the following theorem.

Theorem 2.

For the phase velocities $c = c_r$ of non-singular neutral modes

$$a + \frac{|U''|_{min} - \sqrt{|U''|^2_{max} + 4(\alpha^2+k^2)N^2_{max}}}{2(\alpha^2 + k^2)} \leq c_r \leq b +$$

$$+ \frac{|U''|_{max} + \sqrt{|U''|^2_{max} + 4(\alpha^2+k^2)N^2_{max}}}{2(\alpha^2 + k^2)} \qquad (11)$$

From equation (6), we see that a necessary condition for the existence of non-singular neutral modes is that $U''^2+4\alpha^2N^2 > 0$ atleast once in the flow domain. But, Miles (1961) has shown that if $N^2(y) < 0$ throughout the flow domain then non-singular neutral modes do not exist. However, for homogeneous flows, our result imply that non-singular neutral modes do not exist if $U(y)$ is linear.

3. A Simple Illustration.

For $U \equiv 0$ and $\rho = \rho_o e^{-\beta y}$, where $\beta > 0$ is a constant, Rayleigh (1883) showed that the non-singular neutral modes are given by

$$c_r = \pm\left[\frac{N^2}{k^2 + n^2\alpha^2}\right]^{\frac{1}{2}}, \; n = 1, 2, \ldots \qquad (12)$$

where $N^2 = g\beta$ = positive constant. It follows from equation (12) that

$$\frac{-N}{\alpha} < c_r < \frac{N}{\alpha} \tag{13}$$

and this is precisely what we get from (10) in the present case.

For the present case, equation (11) becomes

$$-\left[\frac{N^2}{\alpha^2+k^2}\right]^{\frac{1}{2}} \leq c_r \leq \left[\frac{N^2}{\alpha^2+k^2}\right]^{\frac{1}{2}} \tag{14}$$

and we see that these bounds coincide with the exact values of c_r given by (12) when $n = 1$.

4. Concluding Remarks.

We have shown that the wave-velocity of non-singular neutral modes of the Taylor-Goldstein problem are bounded. Hence, for the Taylor-Goldstein problem, the wave-velocity of any normal mode is bounded. Furthermore, the bounds obtained in this chapter are the best possible bounds as illustrated. This boundedness of the phase velocity of non-singular neutral modes is a property of incompressible fluids only, as in the next chapter we show that they may be unbounded for compressible shear flows.

CHAPTER - 5

STABILITY OF GASDYNAMIC SHEAR FLOWS

1. Introduction.

Some general stability characteristics of gasdynamic shear flows have been obtained by Lees and Lin (1946). Then, Eckart (1963), Blumen (1970) and Chimonas (1970) initiated studies on the general aspects of the stability of parallel shear flows of an inviscid fluid with particular reference to the effects of compressibility on the mechanism of instability. The extension of the Rayleigh stability criterion and Howard's semicircle theorem to compressible flows, obtained by Lees and Lin (1946) and Eckart (1963) respectively, have each been rederived by Blumen (1970). The Miles' theorem and Howard's estimate for the growth rate of an unstable mode for stratified shear flows have been extended by Chimonas (1970) to stratified compressible shear flows. Dandapat and Gupta (1977) have studied the stability of magnetogasdynamic shear flows.

In this chapter, we follow the work of Blumen (1970) and obtain general stability results. We have shown that shear free compressible flows support supersonic waves but not subsonic waves. These supersonic waves are non-singular neutral modes whose phase velocities do not lie inside any bounded region in the $c_r c_i$-plane. We have found an instability region for subsonic disturbances which depend on the Mach number, wave number and

depth of the fluid layer. For unbounded flows, we have rederived Hoiland's estimate for the growth rate of an unstable mode and have also found a sufficient condition for stability to supersonic disturbances.

2. Basic Equations.

In this chapter, we use the stability equations in their non-dimensional form so that the Mach number characterises the compressibility. Also, we use the stability equation in terms of the perturbation pressure. Since, these were not introduced in the first chapter, we shall do it here.

We shall consider the linear stability of the basic plane parallel flow $U(y)$, of an ideal gas moving in the x'-direction, with transverse variations along the y'-axis. For simplicity, the basic thermodynamic state is assumed to be constant and is characterized by the sound speed

$$a_* = \frac{\nu p}{\rho} \tag{1}$$

where p and ρ are the pressure and density respectively and ν denotes the ratio of specific heats. Superposed on this basic state are small disturbances in the (x', y') components of velocity, (u', v'), and pressure p'.

Non-dimensionalization is carried out by introducing a velocity scale V and a length scale L, which are characteristic of the transverse variations of the basic current U. We then define the dimensionless coordinates, time, velocities

and pressure as

$$\left.\begin{aligned} (x,y) &\doteq (x',y')/L\,, \quad t \doteq \frac{t'V}{L}\,, \\ \bar{U} &\doteq U/V, \quad (u,v) \doteq (u',v')/V, \\ \pi &= p'/\rho V^2. \end{aligned}\right] \qquad (2)$$

The Mach number is $M \equiv \frac{V}{a_*}$. Then the basic system of linearized inviscid equations becomes (e.g. Betchov and Criminale, 1967)

$$u_t + \bar{U}u_x + v\,\bar{U}_y = -\,\pi_x \qquad (3)$$

$$v_t + \bar{U}v_x = -\,\pi_y \qquad (4)$$

$$M^2(\pi_t + \bar{U}\,\pi_x) + u_x + v_y = 0 \qquad (5)$$

where (3) and (4) are the momentum equations and the equations of mass continuity and entropy conservation have been combined to yield equation (5).

Each wave disturbance will be represented in the form

$$q = \hat{q}(y)\exp\left[ik(x-ct)\right] \qquad (6)$$

where q is u,v or π, k is the x-wave number and $c = c_r + ic_i$ is the complex phase velocity. The stability problem is to determine the complex eigenvalues c under the conditions that the wave disturbance (6) satisfies the linear equations and

the boundary conditions separately. Instability corresponds to $c_i > 0$ and consequently an exponential growth of the wave disturbance at the rate of $k\,c_i$.

We shall make use of the differential equations for the amplitudes $\hat{v}$ and $\hat{\pi}$ respectively of transverse velocity component and pressure . These equations may be obtained from (3), (4), (5) and (6) in the form (after dropping the Carret)

$$\left[\frac{(\bar{U}-c)v' - \bar{U}'v}{1-M^2(\bar{U}-c)^2}\right]' - k^2(\bar{U}-c)v = 0 \qquad (7)$$

and

$$(\bar{U}-c)\pi'' - 2\bar{U}'\pi' + k^2(\bar{U}-c)\left[1 - M^2(\bar{U}-c)^2\right]\pi = 0 \qquad (8)$$

where primes denote differentiation w.r.t. y.

Following Lees and Lin (1946), the disturbances associated with (7) or (8) can be classified as subsonic, sonic or supersonic depending on whether the relative phase speed $|\bar{U}-c|$ is less than, equal to or greater than the inverse of the Mach number

$$\text{i.e. } |\bar{U}-c| \lesseqgtr M^{-1} \qquad (9)$$

The subsonic disturbances are the counterpart of the so-called inertial modes, which are solutions of (7) in the limiting case $M = 0$ (the Rayleigh stability equation). The physical significance of the sonic disturbances is apparently not clear

and will not be considered further. Finally, supersonic disturbances correspond to compression or sound waves.

On rigid boundaries the normal velocity must vanish. Thus, from (4), we have

$$v = \pi' = 0 \quad (y = y_1, y_2) \tag{10}$$

If the fluid is unbounded, then the boundary conditions become

$$\pi = v = 0 \quad (y \to \pm \infty) \tag{11}$$

3. Subsonic and Supersonic Waves.

In the absence of basic shear ($\bar{U} \doteq 0$) (7) becomes

$$(-c)\left[\frac{v''}{1-M^2c^2} - k^2 v\right] = 0 \tag{12}$$

Now, from Howard's semicircle theorem (Blumen, 1970), it is seen that $\bar{U} \doteq 0$ is stable. Therefore (12) with (10) can have only real eigenvalues.

Lemma 1.

In the absence of basic shear, subsonic waves do not exist.

Proof.

Multiplying (12) by v and integrating over (y_1, y_2) gives after an integration by parts using (10)

$$(-c) \int \frac{v'^2}{1-M^2c^2} + k^2 v^2 = 0 \tag{13}$$

The limits of integration and the infinitesimal length dy are dropped throughout for convenience.

It we consider only subsonic waves, then $1-M^2c^2 > 0$ and we see from (13) that $c = 0$.

This proves the lemma.

Lemma 2.

In the absence of basic shear, supersonic waves do exist.

Proof.

For $c \neq 0$ and $1-M^2c^2 < 0$, equation (12) may be written as

$$v'' + k^2(M^2c^2-1)v = 0 \tag{14}$$

Now the function

$$v = \sin\left[k\left(\sqrt{M^2c^2-1}\right)(y-y_1)\right] \tag{15}$$

satisfy the equation (14) and also the boundary condition (10) whenever

$$k\sqrt{M^2c^2-1} = \frac{n\pi}{(y_2-y_1)} \quad (n = 1,2,\ldots).$$

That is, whenever

$$c^2 = \frac{1}{M^2}\left\{1 + \frac{n^2\pi^2}{k^2(y_2-y_1)^2}\right\} \tag{16}$$

for $n = 1,2,3,\ldots$.

This proves the lemma.

Combining lemmas 1 - 2, we have the following theorem.

Theorem 1.

Shear free basic flow ($\bar{U} \equiv 0$) supports supersonic waves, but do not support subsonic waves.

Remark 1.

Rayleigh (1883) has shown that shear free basic flow supports internal gravity waves (in the presence of gravity). Here, we have shown that compressibility also introduce new waves in shear free flows and that these waves are supersonic waves.

Remark 2.

For incompressible shear flows, we showed in the last chapter that the wave velocities of non-singular neutral modes are bounded. But, in the presence of compressibility, this result is not true. An example is given by the supersonic waves discussed in lemma 2, which are non-singular neutral modes and their wave velocities given by (16) are unbounded.

4. Instability Region for Subsonic Disturbances.

Eckart (1963) and Blumen (1970) have shown that the complex wave velocity c for any unstable mode lie inside certain semicircle in the upper half of the $c_r c_i$-plane. But their semicircle does not involve the Mach number and hence is the same as that of incompressible shear flows. Considering

only subsonic disturbances, we improve upon this result by finding an instability region which depends on the Mach number and lies inside the semicircle of Eckart (1963). Interestingly, this new instability region depends also on the wave number and depth of the fluid layer.

Theorem 2.

The complex wave velocity c for any unstable subsonic mode must lie inside the semicircle type region

$$(c_r - \frac{a+b}{2})^2 + c_i^2 + \frac{M^2\pi^2(1 - \frac{5}{4}M^2(b-a)^2)^2 c_i^4}{4[\pi^2 + k^2(y_2-y_1)^2(1-\frac{5}{4}M^2(b-a)^2)^2]} \leq (\frac{b-a}{2})^2 \tag{17}$$

Proof.

For $c_i > 0$, the transformation

$$v = (\bar{U}-c)F \tag{18}$$

reduces the original system (7) and (10) to the system

$$\left[\frac{(\bar{U}-c)F'}{1 - M^2(\bar{U}-c)^2}\right]' - k^2(\bar{U}-c)^2F = 0 \tag{19}$$

and $\quad F = 0 \quad$ at $\quad y = y_1, y_2 \qquad (20)$

Following the standard procedure (Howard, 1961), we can obtain the inequality

$$[(c_r - \frac{a+b}{2})^2 + c_i^2 - (\frac{b-a}{2})^2] \int Q$$

$$+ \int \frac{M^2|\bar{U}-c|^4|F'|^2}{|1-M^2(\bar{U}-c)^2|^2} \leq 0 \qquad (21)$$

$$\text{where } Q = \frac{|F'|^2}{|1-M^2(\bar{U}-c)^2|^2} + k^2|F|^2, \quad a \leq \bar{U}(y) \leq b, \qquad (22)$$

Dropping the last term which is positive, one can obtain the semicircle theorem. But, that is precisely the term which involves the compressibility effects explicitly. So, we shall find a lower and relevant bound for the term, though only for subsonic disturbances.

For subsonic disturbances, we have $M|\bar{U}-c| < 1$.

Therefore $|1-M^2(\bar{U}-c)^2| \leq 1+M^2|\bar{U}-c|^2 < 2$,

$$\text{Therefore } \int \frac{M^2|\bar{U}-c|^4|F'|^2}{|1-M^2(\bar{U}-c)^2|^2} \geq \frac{M^2c_i^4}{4} \int |F'|^2 \qquad (23)$$

From the semicircle theorem, we have, for $c_i > 0$,

$$|\bar{U}-c|^2 = (\bar{U}-c_r)^2 + c_i^2 < (b-a)^2 + (\frac{b-a}{2})^2 = \frac{5}{4}(b-a)^2.$$

Therefore, for subsonic modes

$$|1-M^2(\bar{U}-c)^2| \geq |1-M^2|\bar{U}-c|^2|$$

$$\geq 1 - \frac{5}{4} M^2(b-a)^2$$

Therefore, $\int Q \leq \int \{\frac{|F'|^2}{[1 - \frac{5}{4} M^2(b-a)^2]^2} + k^2|F|^2\}$ (24)

Since by the semicircle theorem

$(c_r - \frac{a+b}{2})^2 + c_i^2 - (\frac{b-a}{2})^2 \leq 0$, we have after using (23) and (24)

$$[(c_r - \frac{a+b}{2})^2 + c_i^2 - (\frac{b-a}{2})^2] \int [\frac{|F'|^2}{(1 - \frac{5}{4} M^2(b-a)^2)^2} + k^2|F|^2]$$
$$+ \frac{M^2c_i^4}{4} \int |F'|^2 \leq 0 \qquad (25)$$

Now, by using the Rayleigh-Ritz inequality

$$\int |F|^2 \leq \frac{(y_2-y_1)^2}{\pi^2} \int |F'|^2$$

in (25), we get after dropping $\int |F'|^2$

$$[(c_r - \frac{a+b}{2})^2 + c_i^2 - (\frac{b-a}{2})^2][\frac{1}{(1 - \frac{5}{4} M^2(b-a)^2)^2} + \frac{k^2(y_2-y_1)^2}{\pi^2}]$$
$$+ \frac{M^2c_i^4}{4} \leq 0. \qquad (26)$$

Equation (17) follows from this.

For incompressible fluids $M \equiv 0$ and the above result reduces to Howard's semicircle theorem. Furthermore, if either $k^2 \to \infty$ or $(y_2-y_1) \to \infty$, then also, the above region reduces to

the semicircle. But, for finite values of k^2 and $(y_2-y_1)^2$ and for non-zero, M, the instability region gets reduced.

5. Stability Analysis For Unbounded Flows.

If $c_i \neq 0$, and $\bar{U}$ is finite, then (8) may be divided by $(\bar{U}-c)^3$, with the result that

$$[(\bar{U}-c)^{-2}\pi']' - k^2[(\bar{U}-c)^{-2} - M^2]\pi = 0 \tag{27}$$

Multiplication of (27) by π^* and integration over $(-\infty,\infty)$ and application of (11) yields

$$\int [(\bar{U}-c)^{-2}|\pi'|^2 + k^2\{(\bar{U}-c)^{-2}-M^2\}|\pi|^2] = 0 \tag{28}$$

Following the standard procedure (Howard, 1961) we obtain

$$[(c_r - \frac{a+b}{2})^2 + c_i^2 - (\frac{b-a}{2})^2]\int \bar{Q} + k^2M^2 \int |\pi|^2 \leq 0 \tag{29}$$

where $\bar{Q} = |\bar{U}-c|^{-4}(|\pi'|^2 + k^2|\pi|^2)$ (30)

Since $k^2M^2 \geq 0$ and $\bar{Q} > 0$, (29) gives the semicircle theorem.

As noted by Blumen (1970), we see from (29) that increasing values of (kM) plays the same role in reducing the allowable range of unstable modes as increasing values of the Brunt-Väisällä frequency in Howard's result.

Since we consider only unstable modes, the transformation

$$G = (\bar{U}-c)^{-1/2}\pi \tag{31}$$

is well defined. Use of this transformation in (8) results in the following equation in G :

$$[(\bar{U}-c)^{-1} G']' + \frac{1}{2}[(\bar{U}-c)^{-2}\bar{U}']' G - \frac{1}{4}\bar{U}'^{2}(\bar{U}-c)^{-3}G$$

$$+ k^2(\bar{U}-c)^{-1} G + k^2M^2(\bar{U}-c)G = 0 \qquad (32)$$

and (11) gives $G(y) = 0 \ (y \to \pm \infty)$. (33)

If (32) is multiplied by G and integrated over the flow domain, we get after integration by parts whenever necessary,

$$\int (\bar{U}-c)^{-1}[|G'|^2+k^2|G|^2] + \frac{1}{2}\int (\bar{U}-c)^{-2}\bar{U}'(GG^*)'$$

$$+ \int \frac{\bar{U}'^2}{4}(\bar{U}-c)^{-3}|G|^2 - \int k^2M^2(\bar{U}-c)|G|^2 = 0 \qquad (34)$$

The imaginary part of this gives

$$\int \left\{\frac{|G'|^2}{|\bar{U}-c|^2} + \frac{(\bar{U}-c_r)\bar{U}'(GG^*)'}{|\bar{U}-c|^4} + \frac{\bar{U}'^2(\bar{U}-c_r)^2|G|^2}{|\bar{U}-c|^6}\right\}$$

$$+ \int \frac{k^2|G|^2}{|\bar{U}-c|^2} + \int k^2M^2|G|^2 - \int \frac{\bar{U}'^2|G|^2}{4|\bar{U}-c|^4} = 0 \qquad (35)$$

Considering the first integral, we see that

$$\frac{|G'|^2}{|\bar{U}-c|^2} + \frac{(\bar{U}-c_r)\bar{U}'(GG^*)'}{|\bar{U}-c|^4} + \frac{\bar{U}'^2(\bar{U}-c_r)^2|G|^2}{|\bar{U}-c|^6}$$

$$= \left[\frac{|G'|}{|\bar{U}-c|} - \frac{|\bar{U}'|\,|\bar{U}-c_r|\,|G|}{|\bar{U}-c|^3}\right]^2$$

$$+ \left[\frac{2|\bar{U}'|\ |\bar{U}-c_r|\ |G|\ |G'|}{|\bar{U}-c|^4} + \frac{(\bar{U}-c_r)\ \bar{U}'\ (GG^*)'}{|\bar{U}-c|^4}\right]. \tag{36}$$

Concerning the second square bracket, we observe that

$$\frac{(\bar{U}-c_r)\ \bar{U}'(GG^*)'}{|\bar{U}-c|^4} \leq \frac{|\bar{U}-c_r|\ |\bar{U}'|\ |(GG^*)'|}{|\bar{U}-c|^4}$$

$$\leq \frac{2|\bar{U}-c_r|\ |\bar{U}'|\ |G|\ |G'|}{|\bar{U}-c|^4} \tag{37}$$

Hence the expression on the left hand side of (36) is a non-negative quantity p^2(say), so that (35) can be rewritten as

$$\int \left(p^2 + \frac{k^2|G|^2}{|\bar{U}-c|^2}\right) + \int \left(k^2\ M^2 - \frac{\bar{U}'^2}{4|\bar{U}-c|^4}\right) |G|^2 = 0 \tag{38}$$

This implies that

$$\int \frac{k^2|G|^2}{|\bar{U}-c|^2} \leq \int \frac{\bar{U}'^2\ |G|^2}{4|\bar{U}-c|^4} \tag{39}$$

From this, we get the estimate for the growth rate of any unstable disturbance as

$$k^2\ c_i^2 \leq \frac{1}{4}\ \bar{U}'^2_{max} \tag{40}$$

This estimate was first obtained by Hoiland (1951) for incompressible homogeneous shear flows and can be obtained as a special case, from (4.3) of Chimonas (1970).

Equation (38) also gives the following theorem.

Theorem 3.

A necessary condition for instability is that

$$k^2M^2 < \frac{\bar{U}'^2}{4|\bar{U}-c|^4} \tag{41}$$

atleast once in the flow domain.

This result essentially means only that the flow $\bar{U}(y)$ is stable to short waves.

Now, for supersonic disturbances, we have $|\bar{U}-c)| > M^{-1}$. Hence, for supersonic disturbances, (38) implies that

$$\int \frac{k^2|G|^2}{|\bar{U}-c|^2} \leq \int \left(\frac{1}{4}\bar{U}'^2 - k^2M^{-2}\right) \frac{|G|^2}{|\bar{U}-c|^4} \tag{42}$$

From this, we get the following theorems.

Theorem 4. A sufficient condition for stability of the basic flow $\bar{U}(y)$ to supersonic disturbances is that

$$k^2\ M^{-2} \geq \frac{\bar{U}'^2}{4} \tag{43}$$

throughout the flow domain.

Theorem 5.

For an unstable supersonic disturbance, an estimate for the growth rate is given by

$$k^2 c_i^2 \leq \left(\frac{\bar{U}'^2}{4} - k^2 M^{-2}\right)_{max} \tag{44}$$

The estimate (44) is an improvement over the estimate (40).

6. Concluding Remarks.

We have shown that shear free basic flow ($\bar{U} \equiv 0$) support supersonic waves but do not support any subsonic wave. These supersonic waves are non-singular neutral modes whose wave velocities are not bounded. For subsonic disturbances the instability region given by the semicircle theorem is further improved. The new instability region lies inside the semicircle and depends on the Mach number, wave number and depth of the fluid layer. For unbounded flows, we have rederived Hoiland's estimate for growth rate of any unstable mode. Furthermore, a sufficient condition for stability to supersonic disturbances and an improved estimate for the growth rate of an unstable supersonic mode are also given.

The sufficient condition for stability to supersonic disturbances of an unbounded shear flow is similar to Miles' (1961) theorem for stratified incompressible shear flows. Also, as remarked earlier, the existence of supersonic waves in shear free basic flow is similar to the existence of internal gravity waves. These results seem to imply an analogy between compressibility and density stratification under gravity. This point needs further investigation.

CHAPTER - 6

STABILITY OF STRATIFIED COMPRESSIBLE SHEAR FLOWS

1. Introduction.

In this chapter, we study the linear stability of compressible shear flows in the presence of gravity. The stability equations have already been derived in the first chapter. Due to the mathematical complexity of this problem, no systematic study of this problem has been made so far. Eckart (1963) extended Howard's semicircle theorem to stratified compressible shear flows. Later, Chimonas (1970) extended Miles' theorem and Howard's estimate for the growth rate of an unstable mode, to compressible flows. These results were not simply found and needed considerable ingenity.

For incompressible stratified shear flows, Kochar and Jain (1979) have shown that the complex wave velocities of unstable modes lie inside a certain semiellipse, rather than Howard's semicircle, whose minor axis depends upon stratification. This result has further been generalized by Jain and Kochar (1983 and Makov and Stepanyants (1984). Corresponding results do not exist for compressible flows. Furthermore, nothing is known regarding the role of curvature of the basic velocity profile on the stability of stratified compressible shear flows. In this chapter, we attempt to fill these gaps.

First, we modify Chimonas's proof of Miles' theorem. This enables us to improve upon the instability region given by the semicircle theorem for subsonic disturbances and for a class of supersonic disturbances. The instability region for subsonic disturbances, which is given in this chapter, depends not only on stratification but also on the wave number and the depth of the fluid layer. Furthermore, this region reduces to the line $c_i = 0$ when $U'_{min} \neq 0$ and $J_o \to \frac{1}{4}$ - in accord with Miles' theorem.

To study the curvature effects on stability of stratifie compressible shear flows, we make use of an approximation due to Shivamoggi (1977). Under this approximation, we assume that $\frac{1}{a^2} << 1$ and $c_i << 1$ so that their product can be neglected in comparison to unity, in the stability equations. Under this approximation and for subsonic modes we have found an instabilit criterion and estimates for growth rate, all involving the curvature of the basic velocity profiles. Also, we could extend many standard results of incompressible flow theory.

2. <u>Modified Proof of Miles' Theorem.</u>

The stability equation in terms of G is

$$\left[\frac{r(U-c)G'}{1-\frac{(U-c)^2}{a_*^2}}\right]' - \frac{1}{2}\left[\frac{rU'}{1-\frac{(U-c)^2}{a_*^2}}\right]' G - \frac{rU'^2 G}{(4(U-c)\left(1-\frac{(U-c)^2}{a_*^2}\right)} \cdot$$

$$- rk^2(U-c)G + \frac{rN^2 G}{(U-c)} = 0 \qquad (1)$$

and the boundary conditions are

$$G(y_1) = 0 = G(y_2) \tag{2}$$

For an unstable mode, the imaginary part of the equation, obtained by multiplying (1) by G^* and integrating it using (2), gives

$$\int \frac{r}{\left|1 - \frac{(U-c)^2}{a_*^2}\right|^2} \left\{ \left(1 + \frac{|U-c|^2}{a_*^2}\right) |G'|^2 + k^2 \left| 1 - \frac{(U-c)^2}{a_*^2}\right|^2 |G|^2 + \frac{U'^2 \left(1 + \frac{|U-c|^2}{a_*^2}\right) |G|^2}{4a_*^2} - \frac{U'(U-c_r)\,(GG^*)'}{a_*^2} \right\} + \int \left(N^2 - \frac{U'^2}{4}\right) \frac{r\,|G|^2}{|U-c|^2} = 0 \tag{3}$$

At this stage, we deviate from the analysis of Chimonas (1970) and rewrite equation (3) as

$$\int \frac{r}{\left|1 - \frac{(U-c)^2}{a_*^2}\right|^2} \left\{ \left(1 + \frac{(U-c_r)^2}{a_*^2}\right) \left(|G'|^2 + \frac{U'^2\,|G|^2}{4a_*^2}\right) + \frac{c_i^2}{a_*^2} \left(|G'|^2 + \frac{U'^2\,|G|^2}{4a_*^2}\right) - \frac{U'(U-c_r)\,(GG^*)'}{a_*^2} + k^2 \left|1 - \frac{(U-c)^2}{a_*^2}\right|^2 |G|^2 \right\} + \int \left(N^2 - \frac{U'^2}{4}\right) \frac{r\,|G|^2}{|U-c|^2} = 0 \tag{4}$$

Since $(GG^*)' \leq 2|G|\,|G'|$, we have

$$\left(1 + \frac{(U-c_r)^2}{a_*^2}\right) \frac{|U'|\,|G|\,|G'|}{a_*} - \frac{U'(U-c_r)\,(GG^*)'}{a_*^2} \geq 0 \tag{5}$$

Therefore,

$$(1 + \frac{(U-c_r)^2}{a_*^2}) (|G'|^2 + \frac{U'^2|G|^2}{4a_*^2}) - \frac{U'(U-c_r)(GG^*)'}{a_*^2}$$

$$= [\{1 + \frac{(U-c_r)^2}{a_*^2}\}^{1/2} |G'| - \{1 + \frac{(U-c_r)^2}{a_*^2}\}^{\frac{1}{2}} \frac{|U'|\ |G|}{2a_*}]^2$$

$$+ (1 + \frac{(U-c_r)^2}{a_*^2}) \frac{|U'|\ |G|\ |G'|}{a_*} - \frac{U'(U-c_r)(GG^*)'}{a_*^2}$$

≥ 0, (by 5)).

Therefore, for an unstable mode, (4) implies that

$$\int \frac{r}{|1 - \frac{(U-c)^2}{a_*^2}|^2} \{\frac{c_i^2}{a_*^2} (|G'|^2 + \frac{U'^2|G|^2}{4a_*^2}) + k^2|1 - \frac{(U-c)^2}{a_*^2}|\ |G|^2\}$$

$$+ \int (N^2 - \frac{U'^2}{4}) \frac{|G|^2}{|U-c|^2} \leq 0 \qquad (6)$$

This is impossible if $N^2 \geq \frac{U'^2}{4}$ everywhere in $[Y_1, Y_2]$.
This proves the following (Miles') theorem.

<u>Theorem 1.</u>

A sufficient condition for stability is that $N^2 \geq \frac{1}{4} U'^2$ throughout $[Y_1, Y_2]$.

3. <u>Instability Regions.</u>

Now, we shall prove the following theorem.

<u>Theorem 2.</u>

The complex wave velocity c for any unstable supersonic

mode satisfying $c_i^2 > a_*^2$ must lie inside the semiellipse region, in the upperhalf plane given by

$$(c_r - \frac{a+b}{2})^2 + c_i^2 \left[1 + \frac{4J_o}{\left[1 + (1 + \frac{5(b-a)^2}{a_{*min}^2})\sqrt{1 - 4J_o}\right]^2}\right] \leq (\frac{b-a}{2})^2 \quad (7)$$

where $a \leq U(y) \leq b$.

<u>Proof.</u>

The stability equation in terms of F is (Comes from (23), Chapter 1)

$$\left[\frac{r(U-c)^2 F'}{1 - \frac{(U-c)^2}{a_*^2}}\right]' - rk^2(U-c)^2 F + rN^2 F = 0$$

Multiplying this equation by F^* and integrating it over (y_1, y_2) and following the standard procedure, we get the inequality

$$\left[(c_r - \frac{a+b}{2})^2 + c_i^2 - (\frac{b-a}{2})^2\right] \int rQ + \int rN^2 |F|^2 + \int \frac{r\,|U-c|^4\,|F'|^2}{a_*^2\,\left|1 - \frac{(U-c)^2}{a_*^2}\right|^2} \leq 0 \quad (8)$$

where $Q = \frac{|F'|^2}{\left|1 - \frac{(U-c)^2}{a_*^2}\right|^2} + k^2|F|^2$, $a \leq U(y) \leq b$.

For supersonic unstable modes satisfying $\frac{c_i^2}{a_*^2} > 1$, we have from (6)

$$\geq \frac{4\, J_o\, c_i\, E^2}{\left[1 + \left(1 + \frac{5(b-a)^2}{4\, a_{*min}^2}\right)\sqrt{1-4J_o}\right]^2}$$

$$\geq \frac{4\, J_o\, c_i^2 \int r\, Q}{\left[1 + \left(1 + \frac{5(b-a)^2}{4\, a_{*min}^2}\right)\sqrt{1-4J_o}\right]^2} \tag{12}$$

Using the estimate (12) in (8), we get (7).

This proves the theorem.

Theorem 3.

The complex wave velocity c for any unstable subsonic mode must lie inside the semiellipse type region, in the upper half plane given by

$$\left(c_r - \frac{a+b}{2}\right)^2 + c_i^2 + \frac{J_o\, U'^2_{min}\, (\lambda^2 + k^2)\, a_{*min}^2\, c_i^8}{U'^4_{max}\, a_{*max}^6 \left(\frac{1}{4} - J_o\right) \left[\frac{1}{2} + 2\sqrt{\frac{1}{4} - J_o}\right]^2 \left\{1 - \frac{5(b-a)^2}{4\, a_{*min}^2}\right\}^2}$$

$$\leq \left(\frac{b-a}{2}\right)^2 \tag{13}$$

where $\lambda^2 = \dfrac{a_{*min}^2\, r_{min}\, \pi^2}{4a_{*max}^2\, r_{max}(y_2 - y_1)^2}$.

Proof.

Since, for subsonic unstable modes,

$\dfrac{(U-c_r)^2 + c_i^2}{a_*^2} < 1$, we have $\dfrac{c_i^2}{a_*^2} < 1$ in $[y_1, y_2]$. Therefore, for an

unstable subsonic mode, we have from (6)

$$\int \frac{r}{\left|1 - \frac{(U-c)^2}{a_*^2}\right|^2} \left\{\frac{c_i^2}{a_*^2}\left(|G'|^2 + k^2\left|1 - \frac{(U-c)^2}{a_*^2}\right|^2 |G|^2\right)\right\}$$

$$\leq \left(\frac{1}{4} - J_o\right) \int \frac{r\, U'^2\, |G|^2}{|U-c|^2} \tag{14}$$

Let $$E_1^2 = \int \frac{r\, c_i^2\, |U-c|}{a_*^2 \left|1 - \frac{(U-c)^2}{a_*^2}\right|^2} \left\{|F'|^2 + k^2\left|1 - \frac{(U-c)^2}{a_*^2}\right|^2 |F|^2\right\}$$

and $$B_1^2 = \int \frac{r c_i^2\, |U'|^2\, |F|^2}{a_*^2 \left|1 - \frac{(U-c)^2}{a_*^2}\right|^2 |U-c|} .$$

Then, proceeding as in the proof of the previous theorem, we get

$$E_1^2 \leq \frac{a_{*max}^2}{c_i^2}\left[\frac{1}{2} + 2\sqrt{\frac{1}{4} - J_c}\right]^2 B_1^2 \tag{15}$$

Use of Rayleigh-Ritz inequality shows that

$$\frac{\int \frac{r\, c_i^2\, |G'|^2}{a_*^2 \left|1 - \frac{(U-c)^2}{a_*^2}\right|^2}}{\int \frac{c_i^2\, r |G|^2}{a_*^2}} \geq \lambda^2 \tag{16}$$

where $$\lambda^* = \frac{a_{*min}^2}{a_{*max}^2}\, \frac{r_{min}}{r_{max}}\, \frac{\pi^2}{4(y_2-y_1)^2} . \tag{17}$$

Use of (17) in (14) leads to

$$\lambda^2 + k^2 \le \left(\frac{1}{4} - J_o\right) \frac{U'^2_{max}\ a^2_{*max}}{c_i^4} \ . \qquad (18)$$

If $\nu^2 = \int \dfrac{r\,|F'|^2}{\left|1 - \dfrac{(U-c)^2}{a_*^2}\right|^2} \Big/ \int r\,|F|^2$, then use of (18) gives

$$\int \frac{rQ}{r|F|^2} = \nu^2 + k^2 \le \frac{(\nu^2+k^2)\left(\frac{1}{4} - J_o\right) a^2_{*max}\ U'^2_{max}}{(\lambda^2 + k^2)\ c_i^4} \qquad (19)$$

We have, $$\frac{E_1^2}{B_1^2} = \frac{\displaystyle\int \frac{r\,c_i^2\,|U-c|Q}{a_*^2}}{\displaystyle\int \frac{r\,c_i^2\,|U'|^2\,|F|^2}{a_*^2\left|1 - \dfrac{(U-c)^2}{a_*^2}\right|^2 |U-c|}}$$

$$\ge \frac{c_i^2\ a^2_{*min}\ (\nu^2 + k^2)}{U'^2_{max}\ a^2_{max}\ \left(1 - \dfrac{5(b-a)^2}{4\ a^2_{*min}}\right)^2}$$

which can be rewritten as

$$(\nu^2+k^2) \le \frac{U'^2_{max}\ a^2_{*max}\ \left(1 - \dfrac{5(b-a)^2}{4\ a^2_{*min}}\right)^2}{c_i^2\ a^2_{*min}} \frac{E_1^2}{B_1^2} \ .$$

Using the inequality (15), we get

$$\nu^2 + k^2 \le \frac{U'^2_{max}\ a^4_{*max}\ \left(1 - \dfrac{5(b-a)^2}{4\ a^2_{*min}}\right)^2 \left[\frac{1}{2} + 2\sqrt{\frac{1}{4} - J_o}\right]^2}{c_i^4\ a^2_{*min}} \ . \qquad (20)$$

Using this in (19), we have

$$\frac{\int r\Omega}{\int r|F|^2} \leq \frac{U'^4_{max}\ a^6_{*max}\ \left(1 - \frac{5(b-a)^2}{4\ a^2_{*min}}\right)^2 \left[\frac{1}{2} + 2\sqrt{\frac{1}{4} - J_o}\right]^2 \left(\frac{1}{4} - J_o\right)}{c_i^8\ a^2_{*min}\ (\lambda^2+k^2)} . \tag{21}$$

Therefore, $\int rN^2|F|^2 \geq J_o\ U'^2_{min} \int r|F|^2$

$$\geq \frac{J_o\ U'^2_{min}\ (\lambda^2+k^2) a^2_{*min}\ c_i^8 \int r\Omega}{U'^4_{max}\ a^6_{*max}\ \left(\frac{1}{4} - J_o\right)\left[\frac{1}{2} + 2\sqrt{\frac{1}{4} - J_o}\right]^2 \left(1 - \frac{6(b-a)^2}{4a^2_{*min}}\right)^2} . \tag{22}$$

Use of this in (8), gives (13).

This proves the theorem.

In these two theorems, we have reduced the instability region given by Howard's semicircle. The reduced regions depend on stratification through the minimum Richardson number J_o. The second result for subsonic disturbances incorporate, not only the stratification but also the wave number and the depth of the fluid layer. Furthermore, when $U'_{min} \neq 0$, $J_o \to \frac{1}{4}^-$ implies $c_i \to 0^+$ in accord with Miles' theorem.

4. Curvature Effects.

So far, there is no result which throws light on the role of curvature of the basic velocity profile on the stability of stratified compressible shear flows. However, if we make use of an approximation due to Shivamoggi (1977), we get results

illustrating the role of curvature on the stability of stratified compressible shear flows.

The stability equation is

$$\left[\frac{rH'}{1-\frac{(U-c)^2}{a_*^2}}\right]' - \left[\frac{rU'}{1-\frac{(U-c)^2}{a_*^2}}\right]' \frac{H}{(U-c)} - rk^2H + \frac{rN^2H}{(U-c)} = 0 \qquad (23)$$

and the boundary conditions are

$$H(y_1) = 0 = H(y_2) \qquad (24)$$

Following Shivamoggi (1977), we shall assume that $\frac{1}{a_*^2} \ll 1$ and $c_i \ll 1$ so that their product can be neglected in comparison to unity. Under this approximation, the stability equation (23) becomes

$$\left[\frac{rH'}{1-\frac{(U-c_r)^2}{a_*^2}}\right]' - \left[\frac{rU'}{1-\frac{(U-c_r)^2}{a_*^2}}\right]' \frac{H}{(U-c)} - rk^2H + \frac{rN^2H}{(U-c)^2} = 0 \qquad (25)$$

Now, we shall prove the following theorems for subsonic disturbances, for which $1 - \frac{(U-c_r)^2}{a_*^2} > 0$.

Theorem 4.

A necessary condition for instability to subsonic disturbances is

$$\left[\frac{rU'}{1-\frac{(U-c_r)^2}{a_*^2}}\right]' - \frac{2rN^2(U-c_r)}{(U-c_r)^2 + c_i^2} = 0 \qquad (26)$$

atleast once in (y_1, y_2).

Proof.

Multiplying (25) by H^* and integrating it over (y_1, y_2) using (24), we get

$$\int \frac{r\,|H'|^2}{\left[1 - \frac{(U-c_r)^2}{a_*^2}\right]} + \int rk^2|H|^2 + \int \left[\frac{rU'}{1 - \frac{(U-c_r)^2}{a_*^2}}\right]' \frac{|H|^2}{(U-c)}$$

$$- \int \frac{rN^2 \cdot |H|^2}{(U-c)^2} = 0 \qquad (27)$$

For an unstable subsonic mode, the imaginary part of it gives

$$\int \left\{ \left[\frac{rU'}{1 - \frac{(U-c_r)^2}{a_*^2}}\right]' - \frac{2rN^2(U-c_r)}{(U-c_r)^2 + c_i^2} \right\} \frac{|H|^2}{|U-c|^2} = 0 \; . \qquad (28)$$

The theorem follows from this.

It is interesting to note that when $\frac{1}{a_*^2} = 0$, this result reduces to Synge's criterion (1933). Now, letting

$L = \frac{1}{r}\left[\frac{rU'}{1 - \frac{(U-c_r)^2}{a_*^2}}\right]'$, we have the following theorem.

Theorem 5.

A necessary condition for instability to subsonic disturbances is that

$$c_i^2 < \left(\frac{N^2}{L^2}\right)_{max} \tag{29}$$

whenever L has a definite sign over (y_1, y_2).

Proof.

Equation (28) can be rewritten as

$$\int \{[L(U-c_r)-N]^2 + (L^2 c_i^2 - N^2)\} \frac{r|H|^2}{|U-c|^4} = 0 \tag{30}$$

This implies that a necessary condition for instability is that

$$L^2 c_i^2 - N^2 < 0 \tag{31}$$

atleast once in the flow domain.

The theorem follows from this. For incompressible stratified shear flows, the corresponding result was obtained by Synge (1933) and later, independently, by Yih (1959).

Theorem 6.

An estimate for the growth rate of an unstable subsonic mode is given by

$$k^2 c_i^2 \leq \frac{\dfrac{\left|\left(\dfrac{rU'}{(U-c_r)^2}\right)'\right|_{max} (b-a) + (rN^2)_{max}}{1 - \dfrac{(U-c_r)^2}{a_*^2}}}{r_{min}\left(1 + \dfrac{\pi^2}{k^2(y_2-y_1)^2}\right)} \tag{32}$$

Proof.

The real part of (27) gives

$$\int \left\{ \frac{r|H'|^2}{\left[1 - \frac{(U-c_r)^2}{a_*^2}\right]} + rk^2|H|^2 \right\} = - \int \left[\frac{rU'}{1 - \frac{(U-c_r)^2}{a_*^2}} \right]' \frac{(U-c_r)\ |H|^2}{|U-c|^2}$$

$$+ \int \frac{rN^2((U-c_r)^2 - c_i^2)}{|U-c|^4}\ |H|^2$$

$$\leq \int \left| \left[\frac{rU'}{1 - \frac{(U-c_r)^2}{a_*^2}} \right]' \right| \frac{|U-c_r|}{|U-c|^2}\ |H|^2 + \int \frac{rN^2|H|^2}{|U-c|^2} .$$

This gives

$$r_{min} \int (|H'|^2 + k^2|H|^2) \leq \int \left\{ \left| \left(\frac{rU'}{1 - \frac{(U-c_r)^2}{a_*^2}} \right)' \right|_{max} (b-a) + (rN^2)_{max} \right\} \frac{|H|^2}{|U-c|^2} \tag{33}$$

Use of the Rayleigh-Ritz inequality in (33) gives the estimate (32).

All the three theorems obtained using the approximation of Shivamoggi involve the curvature of the basic flow profile.

5. Instability Conditions.

In addition to the above results involving the curvature of the basic velocity profile, we can obtain results generalizing many standard results of incompressible flow theory. Now, we shall generalize the semiellipse and generalized semiellipse theorems of Kochar and Jain (1979, 1983)

and the bound for the growth rate of Makov and Stepamyants (1984).

The transformation $H = (U-c)F$, when used in (25), leads to the equation

$$\left[\frac{r(U-c)^2 F'}{1 - \frac{(U-c_r)^2}{a_*^2}}\right]' - rk^2(U-c)^2F + rN^2F = 0 \tag{34}$$

and the boundary conditions in terms of F are

$$F(y_1) = 0 = F(y_2) \tag{35}$$

Following Howard (1961), it is easy to show that a necessary condition for instability is

$$\left[(c_r - \frac{a+b}{2})^2 + c_i^2 - (\frac{b-a}{2})^2\right] \int r\bar{Q} + \int rN^2 \, |F|^2 \leq 0 \tag{36}$$

where $$\bar{Q} = \frac{|F'|^2}{\left[1 - \frac{(U-c_r)^2}{a_*^2}\right]} + k^2|F|^2 \tag{37}$$

Dropping the last term, which is positive, from equation (36) gives us the semicircle theorem. Finding a lower and relevant estimate of the last term in (36) will lead to the semiellipse theorem. Before proceeding to do that, we shall prove the following theorem.

Theorem 7.

A necessary condition for instability to subsonic disturbances is that

$$J < \frac{1}{4\left[1 - \frac{(b-a)^2}{a_{*min}^2}\right]} \tag{38}$$

atleast at one point in (y_1, y_2).

Proof.

Use of the transformation $G = (U-c)^{1/2} F$ in (34) leads to the equation

$$\left[\frac{r(U-c)G'}{1 - \frac{(U-c_r)^2}{a_*^2}}\right]' - \frac{1}{2}\left[\frac{rU'}{1 - \frac{(U-c_r)^2}{a_*^2}}\right]' G - \frac{rU'^2 G}{4\left[1 - \frac{(U-c_r)^2}{a_*^2}\right](U-c)}$$

$$- rk^2(U-c)G + \frac{rN^2 G}{(U-c)} = 0 \tag{39}$$

and the associated boundary conditions are

$$G(y_1) = 0 = G(y_2) \tag{40}$$

Multiplying (39) by G^* (complex conjugate of G) and integrating it over (y_1, y_2) using (40), we get

$$\int r(U-c)\left\{\frac{|G'|^2}{\left[1 - \frac{(U-c_r)^2}{a_*^2}\right]} + k^2|G|^2\right\} + \frac{1}{2}\int\left[\frac{rU'}{1 - \frac{(U-c_r)^2}{a_*^2}}\right]' |G|^2$$

$$+ \int \frac{rU'^2 |G|^2}{4\left[1 - \frac{(U-c_r)^2}{a_*^2}\right](U-c)} - \int \frac{rN^2|G|^2}{(U-c)} = 0 . \tag{40}$$

The imaginary part of (40) gives

$$\int \left[\frac{r|G'|^2}{\left[1 - \frac{(U-c_r)^2}{a_*^2}\right]} + rk^2|G|^2 \right] = \int \left[\frac{U'^2}{4\left[1 - \frac{(U-c_r)^2}{a_*^2}\right]} - N^2 \right] \frac{r|G|^2}{|U-c|^2} \quad (41)$$

This is impossible if $N^2 \geq \frac{U'^2}{4\left[1 - \frac{(U-c_r)^2}{a_*^2}\right]}$ in $[y_1, y_2]$.

Hence a sufficient condition for stability to subsonic disturbances is

$$N^2 \geq \frac{U'^2}{4\left[1 - \frac{(U-c_r)^2}{a_*^2}\right]} \quad (42)$$

throughout the flow domain.

By the semicircle theorem, for an unstable subsonic mod[e] $a < c_r < b$. But, then

$$\frac{(U - c_r)^2}{a_*^2} \leq \frac{(b-a)^2}{a_{*min}^2} \quad (43)$$

Therefore, $$\frac{U'^2}{4\left[1 - \frac{(b-a)^2}{a_{*min}^2}\right]} \geq \frac{U'^2}{4\left[1 - \frac{(U-c_r)^2}{a_*^2}\right]} \quad (44)$$

Hence, a necessary condition for instability is

$$N^2 < \frac{U'^2}{4\left[1 - \frac{(U-c_r)^2}{a_*^2}\right]} \leq \frac{U'^2}{4\left[1 - \frac{(b-a)^2}{a_{*min}^2}\right]}$$

atleast once in (y_1, y_2).

This proves the theorem.

The above theorem is true only for subsonic modes and that too under an approximation. Furthermore, it is weaker than Miles' theorem proved by Chimonas (1970). However, the method of proof is the standard one (Howard, 1961) and does not invole any ingenius manipulations. So, it helps us to prove the following theorems.

<u>Theorem 8.</u>

The complex wave velocity c for any unstable subsonic mode must lie inside the semiellipse region, in the upperhalf plane given by

$$(c_r - \frac{a+b}{2})^2 + \frac{2c_i^2}{[1 + \sqrt{1 - 4J_o(1 - \frac{(b-a)^2}{a_{*min}^2})}]} \leq (\frac{b-a}{2})^2 \qquad (45)$$

<u>Theorem 9.</u>

The complex wave velocity c for any unstable subsonic mode must lie inside the semiellipse-type region, in the upperhalf plane, given by

$$(c_r - \frac{a+b}{2})^2 + c_i^2 +$$

$$+ \frac{J_o \, U'^2_{min} \, (\lambda^2+k^2) \, c_i^4}{U'^4_{max} [\frac{1}{4(1 - \frac{(b-a)^2}{a_{*min}^2})} J_o][1 - \frac{(b-a)^2}{a_{*min}^2}][\frac{1}{2} + \sqrt{\frac{1}{4} - J_o(1 - \frac{(b-a)^2}{a_{*min}^2}})]^2}$$

$$\leq \left(\frac{b-a}{2}\right)^2 \tag{46}$$

where $$\lambda^2 = \frac{r_{min}\,\pi^2}{r_{max}\,(y_2-y_1)^2} \tag{47}$$

Theorem 10.

An estimate for the growth rate of an unstable subsonic mode is given by

$$k^2 c_i^2 \leq \frac{U'^2_{max}\left[\dfrac{1}{4\left[1-\dfrac{(b-a)^2}{a^2_{*min}}\right]} - J_o\right]}{\left[1+\dfrac{r_{min}\,\pi^2}{r_{max}\,k^2\,(y_2-y_1)^2}\right]}$$

For $\frac{1}{a_*^2} = 0$, theorems 8 and 9 reduce to those of Kochar and Jain (1979, 1983) and theorem 10 reduce to that of Makov and Stepanyants (1984).

For the proofs of these three theorems, one has to proceed generally along the lines of the proofs of theorems 2 and 3, and hence are not repeated here.

6. Concluding Remarks.

In this chapter, we have studied the linear stability of compressible shear flows in the presence of gravity. Due to the extreme mathematical difficulty, not much work has been done in this direction. Following Eckart (1963) and Chimonas (1970), we have attempted to generalize many of the standard

results of stratified in compressible flow theory to the compressible flow case. First of all, we have modified Chimon proof of Miles' theorem. This enables us to reduce the instability regions for subsonic disturbances and for a class of supersonic disturbances. The reduced instability region for subsonic disturbances depend also on the wave number and depth of the fluid layer. It is interesting to note that this region reduces to the line $c_i = 0$ as $J_o \to \frac{1}{4}-$, when $U'_{min} \neq 0$. This is in accord with Miles' theorem and thus Mile theorem has been linked to the instability region.

The role of curvature of the basic velocity profile on the stability of stratified compressible shear flows is also studied. For this, we use an approximation due to Shivamoggi (1977). Under this approximation, we have generalized an instability criterion and an estimate for growth rate, which were given by Synge (1933) for incompressible shear flows. Also, the estimate given in theorem 6 is a generalization of the estimate given by equation 12 and chapter - 3. In addition many standard results of stratified incompressible flow theory have also been generalized.

However, many questions still remain open. For example, the existence of waves in shear free stratified compressible flows has not been proved. Also, the boundedness or unboundedne of the phase velocity of non-singular neutral modesl has not bee proved yet.

BIBLIOGRAPHY

1. Agrawal, S.C. and Agrawal, G.S. (1961)
Hydromagnetic stability of hetrogeneous shear flows.
J. Phy. Soc. Japan 27, 218-223.

2. Acheson, D.J. (1976)
On over-reflexion
J. Fluid Mech., 77, 433-472.

3. Banerjee, M.B. and Jain, R.K. (1971)
A unified instability criterion of hetrogeneous shear flow.
The Math. Stud. XL, 111-119.

4. Banerjee, M.B., Gupta, J. and Gupta, S.K. (1974)
On reducing Howard's semi-circle.
J. Maths. Phy. Sic. 8(5) 475-484.

5. Banerjee, M.B., Shandil, R.G. and Gupta, J. (1978)
On further reducing Howard's semi-circle.
J. Math. Phy. Sci. 12, 1-

6. Banks, W.H.H., Drazin, P.G. and Zaturska, M.B. (1976)
On the normal modes of parallel flow of inviscid stratified fluid.
J. Fluid Mech. 75, 149-171.

7. Betchov, R. and Criminale, W.O. (Jr.) (1967)
Stability of parallel flows
Academic Press.

8. Blumen, W. (1970)
Shear layer instability of an inviscid fluid.
J. Fluid Mech. 40, 769-781.

9. Blumen, W., Drazin, P.G. and Billings, D.F. (1975)
Shear layer instability of an inviscid compressible fluid Part 2.
J. Fluid Mech. 71, 305-316.

10. Chandrasekhar, S. (1961)
Hydrodynamic and hydromagnetic stability.
Oxford Clarendon Press.

11. Chimonas, G. (1970)
The extension of the Miles-Howard theorem to compressible fluids.
J. Fluid Mech. 43, 833-836.

12. Craik, A.D.D. (1972)
Eigenvalue bounds in linear inviscid stability theory.
J. Fluid Mech., 54(4), 657-670.

13. Dandapat, B.S. and Gupta, A.S. (1975)
On the stability of swirling flow in magnetogasdynamics.
Q. Appl. Maths. 182-186.

14. Dandapat, B.S. and Gupta, A.S. (1977)
Stability of magnetogasdynamic shear flow.
Acta Mechanica, 28, 77-83.

15. Drazin, P.G. (1958)
The stability of a shear layer in an unbounded hetrogeneous inviscid fluid.
J. Fluid Mech. 4, 214-224.

16. Drazin, P.G. and Howard, L.N. (1966)
Hydrodynamic stability of parallel flow of inviscid fluid.
Advances in Applied Mechanics, Vol. 9
Academic Press, New York.

17. Eckart, C. (1960)
Hydrodynamics of oceans and atmospheres.
Pergamon Press, New York.

18. Eckart, C. (1963)
Extension of Howard's circle theorem for adiabatic jets.
Phys. Fluids, 6, 1042-1047.

19. Eckhaus, W. (1965)
Studies in nonlinear stability theory.
Springer-Verlag, New York Inc.

20. Engevik, L. (1973)
On the stability of a shear flow in a stratified, incompressible and inviscid fluid, with special emphasis on the Couette flow.
Acta Mechanica, 18, 285-304.

21. Engevik, L. (1974)
The stream function within the critical layer of a shear flow in a stratified and incompressible fluid.
Acta Mechanica, 19, 169-178.

22. Engevik, L. (1975)
The unstable solutions contiguous to the neutral ones of the equation for the shear flows in stratified incompressible and inviscid fluids.
Acta Mechanica, 21, 159-163.

23. Engevik, L. (1978)
On the applicability of Howard's formula.
J. Fluid Mech. 86, 395-400.

24. Engevik, L., Haugan, P.M. and Klemp, S. (1985)
Stability analysis adjacent to neutral solutions of the Taylor-Goldstein equation when Howard's formula breaks down.
J. Fluid Mech., 159, 347-358.

25. Fjortoft, R. (1950)
Application of integral theorems in deriving criteria of stability of laminar flow and for the baroclinic circular vortex.
Geofys. Publ. 17(6), 1-52.

26. Gupta, J.R., Sood, S.K. and Shandil, R.G. (1982)
A unified instability criterion for hetrogeneous shear flows.
J. Math. Phy. Sci., 16(1), 19-27.

27. Goldstein, S. (1931)
On the stability of superposed streams of fluids of different densities.
Proc. R.Soc.London Ser. A. 132, 524-548.

28. Hatanaka, H. (1947)
On the stability of a surface of discontinuity in a compressible fluid.
J. Soc. Sci. Culture, Japan, 2, 3-7.

29. Helmholtz, H. (1868)
On discontinuous movements of fluids.
Phil. Mag. 36, 337-346.

30. Hoiland, E. (1953)
On two-dimensional perturbation of linear flow.
Geofys. Publ. 18(9), 1-12.

31. Howard, L.N. (1961)
Note on a paper of John W. Miles.
J. Fluid Mech. 10(4), 509-512.

32. Howard, L.N. and Gupta, A.S. (1962)
On hydrodynamic and hydromagnetic stability of swirling flows.
J. Fluid Mech. 14, 463-476.

33. Howard, L.N. (1963)
Neutral curves and stability boundaries in stratified flow.
J. Fluid Mech. 16, 333-342.

34. Howard, L.N. (1965)
The number of unstable modes in hydrodynamic stability problems.
J. Mecanique, 3, 435-443.

35. Howard, L.N. and Maslowe, S.A. (1973)
Stability of stratified shear flows.
Boundary layer Met. 4, 511-

36. Howard, L.N. (1973)
On the stability of compressible swirling flow.
Stud. Appl. Math. Vol. LII, 39-43.

37. Huppert, H.E. (1973)
On Howard's technique for perturbing neutral solutions of Taylor-Goldstein equation.
J. Fluid Mech. 57, 361-368.

38. Jain, R.K. and Kochar, G.T. (1983)
Stability of stratified shear flows.
J. Math. Anal. Appl. 96(1), 269-282.

39. Joseph, D.D. (1976)
Stability of fluid motions. Vol. I and II.
Springer-Verlag, New York.

40. Kelvin, W. (1871)
The influence of wind on waves in water supposed frictionless.
Phil. Mag. (4) 42, 368-374.

41. Kochar, G.T. and Jain, R.K. (1979)
Note on Howard's semi-circle theorem.
J. Fluid Mech. 91(3), 489-491.

42. Kochar, G.T. and Jain, R.K. (1979)
On Howard's semi-circle theorem in hydromagnetics.
J. Phy. Soc. Japan, 47(2).

43. Landau, L. (1944)
Stability of tangential discontinuities in compressible fluid.
Akad. Nauk. SSSR, Comptes Rendus (Dok lady), 44, 139-141.

44. Landau, L.D. and Lifshitz, E.M. (1959)
Fluid Mechanics.
Pergamon Press, London.

45. Lees, L. and Lin, C.C. (1946)
Investigation of the stability of the laminar bound
layer in a compressible fluid.
Tech. Notes. NACA. Wash. no. 1115, 83 pp.

46. Lin, C.C. (1955)
The theory of hydrodynamic stability.
Cambridge University Press.

47. Makov, Yu. N. and Stepanyants, Yu. A. (1984)
Note on the paper of Kochar and Jain on Howard's semicircle theorem.
J. Fluid Mech. 140, 1-10.

48. Miles, J.W. (1958)
On the disturbed motion of a vortex sheet.
J. Fluid Mech. 4, 538-552.

49. Miles, J.W. (1961)
On the stability of hetrogeneous shear flows.
J. Fluid Mech., 10, 496-508.

50. Miles, J.W. (1963)
On the stability of hetrogeneous shear flows. Part 2,
J. Fluid Mech. 16, 209-227.

51. Pellacani, C. (1983)
Shear instabilities in stratified fluids; linear theory.
La Rivista del Nuovo Cimento, 6(3), 26 pp.

52. Rayleigh, J.W.S. (1880)
On the stability, or instability, of certain fluid motions.
Proc. Lond. Math. Soc. 9, 57-70.

53. Rayleigh, J.W.S. (1883)
Investigations on the character of the equilibrium of an incompressible fluid of variable density.
Proc. Lond. Math. Soc. 14, 170-177.

54. Sattinger, D.H. (1967)
On the Rayleigh problem in hydrodynamic stability.
SIAM J. Appl. Math. 15, 419-425.

55. Shultz, M.H. (1973)
Spline analysis.
Prentice Hall Inc. Englewood Cliffs. N.J.

56. Shivamoggi, B.K. (1977)
Inviscid theory of stability of parallel compressible flows.
J. de Mecanique 16(2), 227-255.

57. Stuart, J.T. (1971)
Nonlinear stability of parallel flows.
Annual Review Fluid Mech. 3, 347-370.

8. Synge, J.L. (1933)
The stability of hetrogeneous fluids.
Trans. Roy. Soc. Can. 27, 1-18.

9. Taylor, G.I. (1931)
Effect of variation in density on the stability of superposed streams of fluid.
Proc. Roy. Soc. Lon. A, 132-499-523.

O. Thorpe, S.A. (1969)
Neutral eigensolutions of stability equation for stratified shear flow.
J. Fluid Mech. 36(4), 673-683.

1. Tollmien, W. (1935)
Ein allegemeines kriterium der instabilitat laminarer gesch windigkeits verteilungen, Nachr. Ges. Wiss. Gottingen, Math. Phys. Klasse, 50, 79-114.

2. Yih, C.S. (1957)
On stratified flows in a gravitational field.
Tellus, 9, 220-227.

3. Yih, C.S. (1974)
Wave motion in stratified fluids, in "Nonlinear Waves" (S. Leibovich and A.R. Seebass, eds.), 263-281, Cornell University Press, New York.

64. Yih, C.S. (1980)
Stratified flows.
Academic Press, New York.

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